Minimal affinizations and their graded limits

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Jacobi-Trudi formula For a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$,

$$s_\lambda(x) = \det\left(h_{\lambda_i - i + j}(x)\right)_{1 \leq i,j \leq n}.$$

$s_\lambda(x)$: Schur polynomial, $h_k(x)$: complete symm. polynomial.

Translation in the $\mathfrak{sl}_{n+1}$-modules

$\lambda \in P^+$: dom. int. wt $\rightsquigarrow \lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$ by $\lambda_i = \sum_{k \geq i}\langle h_k, \lambda\rangle$

$\mathrm{ch}\, V(\lambda) = s_\lambda(x)$, $\mathrm{ch}\, V(k\varpi_1) = h_k(x)$ $\big(V(\lambda)$: simple $\mathfrak{sl}_{n+1}$-mod.$\big)$

Theorem

$$\mathrm{ch}\, V(\lambda) = \det\left(\mathrm{ch}\, V\big((\lambda_i - i + j)\varpi_1\big)\right)_{1 \leq i,j \leq n}$$

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So $\mathrm{ch}\, V(\lambda) = \det\Big(\mathrm{ch}\, V\big((\lambda_i - i + j)\varpi_1\big)\Big)_{1\leq i,j\leq n}$ holds in type A.

Q. Does this formula hold in other types? **No!**

$$\mathrm{ch}\, V(\lambda) \neq \det\Big(\mathrm{ch}\, V\big((\lambda_i - i + j)\varpi_1\big)\Big)_{1\leq i,j\leq n},$$

when $\mathfrak{g} \neq \mathfrak{sl}_{n+1}$ (though there may be several generalizations.)

Q. When $\mathfrak{g} \neq \mathfrak{sl}_{n+1}$, does $\det\Big(\mathrm{ch}\, V\big((\lambda_i - i + j)\varpi_1\big)\Big)_{1\leq i,j\leq n}$

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In type BD, we have

$$\operatorname{ch} L_q(\lambda) = \det\Big(\operatorname{ch} V\big((\lambda_i - i + j)\varpi_1\big)\Big)_{1\le i,j\le n},$$

where $L_q(\lambda)$ denotes a **minimal affinization** (a special class of f.d. simple $U_q(\mathcal{L}\mathfrak{g})$-modules explained later).

In type C, a similar formula holds:

$$\operatorname{ch} L_q(\lambda) = \det\Big(\sum_{0\le 2k\le \lambda_i - i + j} \operatorname{ch} V\big((\lambda_i - i + j - 2k)\varpi_1\big)\Big)_{1\le i,j\le n}.$$

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1. Definition of minimal affinizations $L_q(\lambda)$
2. Main Theorem (JT formula for $\mathrm{ch}\, L_q(\lambda)$)
3. Proof (Combination of results proved by [N], [Chari-Greenstein], [Sam])

In the proof, **graded limits** ($\mathbb{Z}$-graded $\mathfrak{g} \otimes \mathbb{C}[t]$-modules) are used.

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$\mathfrak{g}$: simple Lie algebra of rank n,

$\mathcal{L}\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$: loop algebra, $\big([x \otimes f, y \otimes g] = [x, y] \otimes fg\big)$

$U_q(\mathcal{L}\mathfrak{g})$: quantum loop algebra/$\mathbb{C}(q)$ $\big(q$-analog of $U(\mathcal{L}\mathfrak{g})\big)$

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$U_q(\mathfrak{g})$: quantum group assoc. with $\mathfrak{g}$ $\big(q$-analog of $U(\mathfrak{g})\big)$

(Note: $\mathfrak{g} = \mathfrak{g} \otimes 1 \subseteq \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] = \mathcal{L}\mathfrak{g}$)

Fact

(1) $\{\text{f.d. simple } \mathfrak{g}\text{-mod.}\} \overset{1:1}{\longleftrightarrow} P^+ \overset{1:1}{\longleftrightarrow} \{\text{f.d. simple } U_q(\mathfrak{g})\text{-mod}\}$

$$V(\lambda) \in \{\ldots\}, \quad \lambda \in P^+, \quad V_q(\lambda) \in \{\ldots\}$$

(2) The cat. of f.d. $\mathfrak{g}$-modules and $U_q(\mathfrak{g})$-modules are semisimple.

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$$V(\lambda) \quad \lambda \quad V_q(\lambda)$$

(each element lying in the set above it: $V(\lambda) \in \{\text{f.d. simple } \mathfrak{g}\text{-mod.}\}$, $\lambda \in P^+$, $V_q(\lambda) \in \{\text{f.d. simple } U_q(\mathfrak{g})\text{-mod}\}$)

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<u>Fact.</u> V: an arbitrary f.d. simple $U_q(\mathcal{L}\mathfrak{g})$-module

$\rightsquigarrow {}^{\exists!}\lambda \in P^+$ s.t. $V \cong V_q(\lambda) \oplus \bigoplus_{\mu<\lambda} V_q(\mu)^{\oplus m_\mu(V)}$ as a $U_q(\mathfrak{g})$-module.

In this case, V is called an **affinization** of $V_q(\lambda)$.

$\{U_q(\mathfrak{g})$-isom. classes of affiniz. of $V_q(\lambda)\}$ $\Leftarrow$ partial order is defined

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Examples of Minimal affinizations

Minimal affinizations for $\mathfrak{g} = \mathfrak{sl}_{n+1}$

When $\mathfrak{g} = \mathfrak{sl}_{n+1}$, $^\exists$alg. hom. $\varphi\colon U_q(\mathcal{L}\mathfrak{g}) \twoheadrightarrow U_q(\mathfrak{g})$ (evaluation map)

(q-analog of the map $\mathcal{L}\mathfrak{g} \twoheadrightarrow \mathfrak{g}\colon x \otimes f \to f(a)x$ for any $a \in \mathbb{C}^\times$)

$\rightsquigarrow \varphi^* V_q(\lambda)$: simple $U_q(\mathcal{L}\mathfrak{g})$-mod. $\Leftarrow$ minimal affinization of $V_q(\lambda)$

($\because \varphi^* V_q(\lambda) \cong V_q(\lambda)$ as a $U_q(\mathfrak{g})$-mod.)

Remark. If $\mathfrak{g} \neq \mathfrak{sl}_{n+1}$, evaluation map **does not** exist.

$\rightsquigarrow$ Most of minimal affinizations are reducible as a $U_q(\mathfrak{g})$-module,

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Kirillov-Reshetikhin modules

Kirillov-Reshetikhin modules $W^{(j)}_{k,a}$ $\big(1 \leq j \leq n, k \in \mathbb{Z}_{\geq 0}, a \in \mathbb{C}(q)\big)$

Properties

- ferimionic character formula
- having crystal bases
- T-system $\Rightarrow$ Monoidal categorification by Hernandez-Leclerc

$$0 \to W^{(i-1)}_{k,aq} \otimes W^{(i+1)}_{k,aq} \to W^{(i)}_{k,a} \otimes W^{(i)}_{k,aq^2} \to W^{(i)}_{k+1,a} \otimes W^{(i)}_{k-1,aq^2} \to 0$$

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Main Theorem

In the sequel, assume that $\mathfrak{g}$ is of type $ABCD$.

Let $\lambda \in P^+$, and let $L_q(\lambda)$ be a minimal affinization of $V_q(\lambda)$.

Theorem

Assume that $\begin{cases} \langle h_n, \lambda \rangle = 0 & \text{if } \mathfrak{g}\text{: type } BC, \\ \langle h_{n-1}, \lambda \rangle = \langle h_n, \lambda \rangle = 0 & \text{if } \mathfrak{g}\text{: type } D, \end{cases}$

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Remark. In type A, this is JT formula since $\mathrm{ch}\, L_q(\lambda) = \mathrm{ch}\, V(\lambda)$.

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Remark. For $k \in \mathbb{Z}_{\geq 0}$, it holds that

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$\lambda \in P^+$: as above. For every $\mu \in P^+$,

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$$\mathrm{ch}\, L_q(\lambda) = \det\left(\mathrm{ch}\, L_q\big((\lambda_i - i + j)\varpi_1\big)\right)_{1\le i,j\le n}.$$

1. In [Nakai-Nakanishi, 06], they have conjectured some formulas for q-characters of $L_q(\lambda)$ (q-character $\stackrel{\text{specialize}}{\to}$ character).
   In fact the specialization of their formula coincides with $\det\left(\mathrm{ch}\, L_q\big((\lambda_i - i + j)\varpi_1\big)\right)_{1\le i,j\le n}$.
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Graded limits

$L_q(\lambda)$: $U_q(\mathcal{L}\mathfrak{g})$-mod./$\mathbb{C}(q) \xrightarrow{q\to 1} L_1(\lambda)$: $\mathcal{L}\mathfrak{g}$-mod./$\mathbb{C}$ (classical limit)

$\xrightarrow{\text{restrict}} L_1(\lambda)$: $\mathfrak{g}[t]$-module $\quad \big(\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t] \subseteq \mathcal{L}\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]\big)$

Fact. $^{\exists}a \in \mathbb{C}^{\times}$ s.t. $\big(\mathfrak{g} \otimes (t+a)^N\big)L_1(\lambda) = 0 \quad (N \gg 0)$

$\rightsquigarrow$ Define an auto. τ_a on $\mathfrak{g}[t]$ by $\tau_a\big(g \otimes f(t)\big) = g \otimes f(t+a)$

$L(\lambda) := \tau_a^*\big(L_1(\lambda)\big)$: **graded limit** of $L_q(\lambda)$ ($\mathbb{Z}$-graded $\mathfrak{g}[t]$-module)

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Proposition (N)

Let $M(\lambda)$ be the $\mathfrak{g}[t]$-module generated by a vector v with relations

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Then the graded limit $L(\lambda)$ is isomorphic to $M(\lambda)$.

A sketch of the proof of this theorem will be given later.

Proposition (Chari-Greenstein, 11)

$$\sum_{(\lambda,s)\in\Gamma(\mu)} (-1)^s \dim \mathrm{Hom}_{\mathfrak{g}}\Big(V(\lambda), \bigwedge^s \mathfrak{g} \otimes V(\mu)\Big) \mathrm{ch}\, M(\lambda) = \mathrm{ch}\, V(\mu),$$

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Proposition (Sam, 14)

Setting $H_\lambda = \det\Big(\mathrm{ch}\, L_q((\lambda_i - i + j)\varpi_1)\Big)_{1\leq i,j\leq n}$,

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Sketch of the proof for $L(\lambda) \cong M(\lambda)$

$M(\lambda)$: the $\mathfrak{g}[t]$-module generated by v with relations

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generalized Demazure module

$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$: affine Lie algebra $\supseteq \mathfrak{g}[t]$

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$\Lambda^1, \dots, \Lambda^k$: dom. int. wts of $\widehat{\mathfrak{g}}$, $\quad w_1, \dots, w_k \in \widehat{W}$

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$$(\because \; L(\mu_i) \hookrightarrow \widehat{V}(\Lambda^i) \text{ for each } i)$$

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We can determine the defining relations of $D(\Lambda^1, \dots, \Lambda^n; w_1, \dots, w_n)$ by the induction on $\sum_i \ell(w_i)$.

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