Noncommutativity between operations of taking tensor product and classical limit of modules over $U_q(\mathcal{L}\mathfrak{g})$

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Notation

$\mathfrak{g}$: simple Lie algebra/$\mathbb{C}$ $\quad (\mathrm{rank}\,\mathfrak{g} = n)$,

$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$: triangular dec.,

$\mathcal{L}\mathfrak{g} := \mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}]$: loop algebra $\quad ([X \otimes f, Y \otimes g] = [X, Y] \otimes fg)$,

$U_q(\mathcal{L}\mathfrak{g})$: quantum loop algebra/$\mathbb{C}(q)$ $\quad \overset{q\text{-deform.}}{\Leftarrow} \; U(\mathcal{L}\mathfrak{g})$

V: f.d. $U_q(\mathcal{L}\mathfrak{g})$-mod. $\quad \overset{\text{“}\lim_{q \to 1}\text{”}}{\Rightarrow} \quad \overline{V}$: f.d. $\mathcal{L}\mathfrak{g}$-mod. (classical limit)

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Precise definition of classical limits

$$\mathcal{A} = \left\{ f(q)/g(q) \in \mathbb{C}(q) \mid g(1) \neq 0 \right\} \subseteq \mathbb{C}(q),$$

$$U_q(\mathcal{L}\mathfrak{g}) \supseteq U_{\mathcal{A}}(\mathcal{L}\mathfrak{g}) := \langle E_i, K_i^{\pm 1}, F_i \rangle_{\mathcal{A}\text{-alg.}},$$

<u>Fact</u> $U_{\mathcal{A}}(\mathcal{L}\mathfrak{g}) \otimes_{\mathcal{A}} \mathbb{C} \twoheadrightarrow U(\mathcal{L}\mathfrak{g})$.

V: fin. dim. $U_q(\mathcal{L}\mathfrak{g})$-mod. Assume that V is ℓ-h.w. module with ℓ-h.w.vector v (i.e. $U_q(\mathcal{L}\mathfrak{n}_+)v = 0$, $U_q(\mathcal{L}\mathfrak{h})v = \mathbb{C}v$, $U_q(\mathcal{L}\mathfrak{n}_-)v = V$), and further assume that $U_{\mathcal{A}}(\mathcal{L}\mathfrak{g})v \subseteq V$ is an $\mathcal{A}$-lattice of V

($\Leftrightarrow$ mild condition on the ℓ-h.w.)

Definition (Chari-Pressley, 01)

$\overline{V} := U_{\mathcal{A}}(\mathcal{L}\mathfrak{g})v \otimes_{\mathcal{A}} \mathbb{C}$: classical limit $\leftarrow \mathcal{L}\mathfrak{g}$-mod.

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problem

$V_1, \dots, V_p$: f.d. $U_q(\mathcal{L}\mathfrak{g})$-mod. ($v_k \in V_k$: ℓ-h.w. vector)

$$\overline{V_1 \otimes \cdots \otimes V_p} = U_{\mathcal{A}}(\mathcal{L}\mathfrak{g})(v_1 \otimes \cdots \otimes v_p) \otimes_{\mathcal{A}} \mathbb{C}$$

$\updownarrow$ not necessarily isomorphic

$$\overline{V_1} \otimes \cdots \otimes \overline{V_p} \cong \big(U_{\mathcal{A}}(\mathcal{L}\mathfrak{g})v_1 \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} U_{\mathcal{A}}(\mathcal{L}\mathfrak{g})v_p\big) \otimes_{\mathcal{A}} \mathbb{C}$$

Q. Can we construct $\overline{V_1 \otimes \cdots \otimes V_p}$ from $\overline{V_1}, \dots, \overline{V_p}$?

We solved this question affirmatively for Kirillov-Reshetikhin modules.

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Notions for the statement of the main theorem

Kirillov-Reshetikhin modules

$W^{i,\ell}(a)$ ($1 \leq i \leq n, \ell \in \mathbb{Z}_{>0}, a \in \mathbb{C}(q)^{\times}$): Kirillov-Reshetikhin mod.

(a distinguished family of f.d. simple $U_q(\mathcal{L}\mathfrak{g})$-mod.)

graded limits

$\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t] \subseteq \mathcal{L}\mathfrak{g}$: current algebra,

φ_c ($c \in \mathbb{C}$): auto. on $\mathfrak{g}[t]$ defined by $\varphi_c(X \otimes f(t)) = X \otimes f(t+c)$.

Fact Assume $a \in \mathcal{A}^{\times}$, and set $c = a(1) \in \mathbb{C}^{\times}$.

Then $\varphi_{-c}^{*}\overline{W^{i,\ell}(a)}$ is $\mathbb{Z}$-graded $\mathfrak{g}[t]$-mod., and independent of a.

$\rightsquigarrow$ write $W^{i,\ell} := \varphi_{-c}^{*}\overline{W^{i,\ell}(a)}$: graded limit of $W^{i,\ell}(a)$.

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fusion product [Feigin-Loktev, '99]

$M_1, \dots, M_p$: cyclic $\mathbb{Z}$-graded $\mathfrak{g}[t]$-mod. ($v_k \in M_k$: generator)

Take $c_1, \dots, c_p \in \mathbb{C}$ (pairwise distinct),

and set $M := \varphi_{c_1}^* M_1 \otimes \cdots \otimes \varphi_{c_p}^* M_p$.

Fact M is generated by $v_1 \otimes \cdots \otimes v_p$ (though not $\mathbb{Z}$-graded).

(Note that usual tensor product $M_1 \otimes \cdots \otimes M_p$ is not cyclic)

Define a filtration $F_{-1}(M) = 0 \subseteq F_0(M) \subseteq \cdots \subseteq F_N(M) = M$ ($N \gg 0$) by $F_k(M) = U(\mathfrak{g}[t])_{\leq k}(v_1 \otimes \cdots \otimes v_p)$, and take the associated graded $\bigoplus_k F_k(M)/F_{k-1}(M) \leftarrow \mathbb{Z}$-graded $\mathfrak{g}[t]$-mod.

This is called the fusion product of $M_1, \dots, M_p$, and denoted by $M_1 * \cdots * M_p$.

fusion prod. = deformed $\otimes$ preserving cyclicity and $\mathbb{Z}$-gradedness fusion product [Feigin-Loktev, '99]

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Main Theorem

Theorem (N)

Assume that a given tensor prod. $W^{i_1,\ell_1}(a_1) \otimes \cdots \otimes W^{i_p,\ell_p}(a_p)$ of KR mod. has its classical limit ($^\exists$sufficient conditions for i_k, ℓ_k, a_k).

(i) If $a_1(1) = \cdots = a_p(1) (=: c)$, then we have a $\mathfrak{g}[t]$-mod. isom.

$$\overline{W^{i_1,\ell_1}(a_1) \otimes \cdots \otimes W^{i_p,\ell_p}(a_p)} \cong \varphi_c^*(W^{i_1,\ell_1} * \cdots * W^{i_p,\ell_p}).$$

(ii) In the general case, we have a $\mathfrak{g}[t]$-mod. isom.

$$\overline{W^{i_1,\ell_1}(a_1) \otimes \cdots \otimes W^{i_p,\ell_p}(a_p)} \cong \bigotimes_{c\in\mathbb{C}^\times} \varphi_c^* \Bigg(\mathop{\Large *}_{k;\, a_k(1)=c} W^{i_k,\ell_k} \Bigg).$$

<u>recall</u> φ_c: auto. on $\mathfrak{g}[t]$ defined by $\varphi_c(X \otimes f(t)) = X \otimes f(t+c)$, $W^{i,\ell}$: the graded limit of $W^{i,\ell}(a)$, $*$: fusion product