An approach to the $X = M$ conjecture using modules over a current algebra

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Plan

1. What is the $X = M$ conjecture?

$$\boxed{\mathbf{1}\text{-dimensional sum } X} \stackrel{?}{=} \boxed{\text{fermionic formula } M} \in \mathbb{Z}[q^{\pm 1}]$$

(crystal basis theory) (Bethe Ansatz)

- Definitions of X, M (with histrical background)

2. Proof in type AD (and partially in BC)

◇ Use modules over a **current algebra** $\mathfrak{g} \otimes \mathbb{C}[t]$

- graded limits of KR modules
- fusion products
- generalized Demazure modules

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State space: $(\mathbb{C}^2)^{\otimes L} = \bigoplus_r W_r$

$$\#\left\{\text{Bethe vectors in } W_r\right\} = \sum_{\substack{m=\{m_j \in \mathbb{Z}_{\geq 0}\}_{j\geq 1} \\ \text{s.t. } 2\sum j m_j = L-r}} \prod_j \binom{p_j + m_j}{m_j}$$

$$\left(p_j = L - 2\sum_k \min\{j,k\} m_k \quad (\textbf{vacancy number}) \right)$$

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The formula can be generalized in type $\mathfrak{sl}_{n+1}$:

$$\left[V(\mu_1\varpi_1)\otimes\cdots\otimes V(\mu_p\varpi_1) : V(\lambda)\right] = \sum_{\substack{m=\{m_j^{(a)}\}_{\substack{1\le a\le n \\ j\ge 1}} \\ \text{s.t. } p_j^{(a)}\ge 0,\ \cdots}} \prod_{a,j} \binom{p_j^{(a)}+m_j^{(a)}}{m_j^{(a)}}.$$

Moreover, the **graded version** also holds!

$$K_{\lambda,\mu}(q) = \sum_m q^{cc(m)} \prod_{a,j} \begin{bmatrix} p_j^{(a)}+m_j^{(a)} \\ m_j^{(a)} \end{bmatrix}_q \in \mathbb{Z}_{\ge 0}[q^{\pm 1}].$$

($K_{\lambda,\mu}(q)$: Kostka polynomial, $\begin{bmatrix} \ \end{bmatrix}_q$: q-binomial coeff.
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They proved that there exists a bijection (KKR bijection)

$$\mathbf{SST}(\lambda,\mu)\text{: s.s. tableaux} \overset{1:1}{\leftrightarrow} \mathbf{RC}(\mu,\lambda)\text{: rigged configurations}$$

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Nongraded version for general type

$\mathfrak{g}$: simple Lie algebra of rank n, $\quad I = \{1, \ldots, n\}$,

$U'_q(\hat{\mathfrak{g}})$: quantum affine algebra (without a degree operator),

$W^{r,\ell}$: Kirillov-Reshetikhin (KR) module ($r \in I$, $\ell \in \mathbb{Z}_{>0}$)

(a family of f.d. simple $U'_q(\hat{\mathfrak{g}})$-modules).

(Nongraded) fermionic formula in general type is stated as follows.

Theorem ([Nakajima, '03], [Hernandez, '06], [DiFrancesco, Kedem, '08])

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$$\mathbb{Z}_{\geq 0}[q^{\pm 1}] \ni \quad X(W, \lambda, q) \quad = \quad M(W, \lambda, q) \quad \in \mathbb{Z}_{\geq 0}[q^{\pm 1}]$$

$M(W, \lambda, q)$ can be defined similarly as in type A_n:

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How to define $X(W, \lambda, q)$?

In type A_n, a Kostka polynomial $K_{\lambda,\mu}(q)$ is expressed as a generating function of **semistandard tableaux**.

In general type, we use **crystal bases** instead.

Theorem ([Kashiwara, '04], [Okado, Schilling, '08])

The KR module $W^{r,\ell}$ has a crystal basis if

- $\ell = 1$, ($\hat{\mathfrak{g}}$: general type, $^\forall r$),
- $\hat{\mathfrak{g}}$: nonexceptional type ($\overset{\text{Def}}{\Leftrightarrow}$ $\mathfrak{g}$: classical), ($^\forall r$, $^\forall \ell$).

$B^{r,\ell}$: the corresponding crystal graph (KR crystal),

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Then the **1-dimensional sum** $X(W, \lambda, q) \in \mathbb{Z}_{\geq 0}[q^{\pm}]$ is defined by

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$$K_{\lambda,\mu}(q)$$

$$\sum q^{cc(m)} \prod \begin{bmatrix} p+m \\ m \end{bmatrix}_q$$

- Consider two sets $\mathbf{SST}(\lambda, \mu)$, $\mathbf{RC}(\mu, \lambda)$ with charges.
- Consider their generating functions.
- Show $\mathbf{SST}(\lambda, \mu) \cong \mathbf{RC}(\mu, \lambda)$ as charged sets.

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Instead of charged sets, we shall use $\mathbb{Z}$**-graded $\mathfrak{g}$-modules**.

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  (graded multiplicity $\left[D : V(\lambda)\right]_q := \sum_{k \in \mathbb{Z}} q^{-k}\left[D_k : V(\lambda)\right]$)
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◦ $\mathfrak{g}[t]$ has a natural $\mathbb{Z}$-grading w.r.t. the degree of t

$\rightsquigarrow L$: $\mathbb{Z}$-graded $\mathfrak{g}[t]$-module $\Rightarrow L$: $\mathbb{Z}$-graded $\mathfrak{g}$-module.

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The work of Ardonne-Kedem

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$\overset{\text{restriction}}{\to}$ $\overline{W^{r,\ell}}$: $\mathfrak{g}[t]$-module (not $\mathbb{Z}$-graded)

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Ex. $\mathfrak{g} = \mathfrak{sl}_{n+1}$

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Next goal

Construct a $\mathbb{Z}$-graded $\mathfrak{g}[t]$-module D s.t. $D \cong L$.

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Notation

In the sequel, assume $\mathfrak{g}$ is of type $ABCD$ for simplicity.

$\hat{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$: affine Lie algebra $\supseteq \mathfrak{g}[t]$,

$\hat{I} := \{0\} \sqcup I$: index set of $\hat{\mathfrak{g}}$,

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Demazure module

For $w = s_{i_1} \cdots s_{i_k} \in \hat{W}$, define a $\hat{\mathfrak{b}}$-submodule $D(w, \Lambda)$ of $\hat{V}(\Lambda)$ by

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- In some cases, $D(w, \Lambda)$ extends to a $\mathfrak{g}[t]$-module. $\Leftarrow$ $\mathbb{Z}$-graded

For $r \in I$, $c_r := \begin{cases} 2 & (\mathfrak{g} : BC,\ \alpha_r : \text{ short root}), \\ 1 & (\text{otherwise}). \end{cases}$ $\Leftarrow$ ${}^\forall r \in I$ in type AD

Theorem ([Chari, Moura, '06])

If $c_r | \ell$, it follows that

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$\vec{w} = (w_1, \dots, w_p)$: seq. of elements of $\hat{W}$,

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(**generalized Demazure module**)

Theorem (N)

If $c_{r_j} | \ell_j$ for all j, it follows that

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where $\vec{w} = (w^{r_1}, \dots, w^{r_p})$, $\vec{\Lambda} = ((\ell_1/c_{r_1})\Lambda_{r'_1}, \dots, (\ell_p/c_{r_p})\Lambda_{r'_p})$.

Recall. L = (fusion product of $L^{r_1,\ell_1}, \dots, L^{r_p,\ell_p}$)

$\vec{w} = (w_1, \dots, w_p)$: seq. of elements of $\hat{W}$,

$\vec{\Lambda} = (\Lambda^1, \dots, \Lambda^p)$: seq. of dominant integral weights of $\hat{\mathfrak{g}}$,

$$D(\vec{w}, \vec{\Lambda}) := \mathcal{F}_{w_1}\Big(\mathbb{C}u_{\Lambda^1} \otimes \cdots \otimes \mathcal{F}_{w_{p-1}}\Big(\mathbb{C}u_{\Lambda^{p-1}} \otimes \mathcal{F}_{w_p}(\mathbb{C}u_{\Lambda^p})\Big)\cdots\Big)$$
$$\subseteq \hat{V}(\vec{\Lambda}) := \hat{V}(\Lambda^1) \otimes \cdots \otimes \hat{V}(\Lambda^{p-1}) \otimes \hat{V}(\Lambda^p).$$

(**generalized Demazure module**)

Theorem (N)

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If $c_{r_j}|\ell_j$ for all j, it follows that

$$L\cong D(\vec{w},\vec{\Lambda}) \Rightarrow \big[L:V(\lambda)\big]_q = \big[D(\vec{w},\vec{\Lambda}):V(\lambda)\big]_q$$

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Plan of the proof

$$X(W, \lambda, q) \quad = \quad \left[D(\vec{w}, \vec{\Lambda}) : V(\lambda)\right]_q$$

$$\parallel$$

$$M(W, \lambda, q) \quad = \quad \left[L : V(\lambda)\right]_q$$

Plan of the proof

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\begin{array}{ccc}
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Generalized Demazure crystal

$\hat{V}_q(\Lambda)$: simple h.w. $U_q(\hat{\mathfrak{g}})$-module $\Rightarrow$ $B(\Lambda)$: crystal basis

$$D(\vec{w},\vec{\Lambda}) \subseteq \hat{V}(\Lambda^1) \otimes \cdots \otimes \hat{V}(\Lambda^p) \curvearrowleft \hat{\mathfrak{g}}$$

$\downarrow$ q-analog $\downarrow$

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$$\left[D(\vec{w},\vec{\Lambda}) : V(\lambda)\right]_q = \sum_{b \in B_\lambda} q^{-\langle \mathrm{wt}(b), d\rangle} \quad (d : \text{degree operator}).$$

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If $c_{r_j}|\ell_j$ for all j ($\Leftrightarrow B^{r_j,\ell_j}$: perfect), there exists a bijection

$$\Psi : B(\vec{w},\vec{\Lambda}) \xrightarrow{\sim} B$$

which preserves weights, commutes with $\tilde{e}_i$ ($i \in I$) and satisfies

$$-\langle d, \mathrm{wt}(b)\rangle = D(\Psi(b)) \quad \text{for } b \in B(\vec{w},\vec{\Lambda}).$$

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Corollary

$\hat{\mathfrak{g}} = A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}$, $W := W^{r_1, c_{r_1}\ell_1} \otimes \cdots \otimes W^{r_p, c_{r_p}\ell_p}$.

If $c_{r_j} | \ell_j$ for all j, we have

$$X(W, \lambda, u) = M(W, \lambda, u).$$

Note that c_r is 1 for all r when $\mathfrak{g}$ is of type AD.

Rem. In the published paper, the corollary is proved for $\mathfrak{g} = AD$ only.

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