Minimal affinizations and their graded limits

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Repsentation theory and Related Topics @ Irako View Hotel

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Introduction

$\boxed{\text{Jacobi-Trudi formula}}$ For a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$,

$$s_\lambda(x) = \det\left(h_{\lambda_i - i + j}(x)\right)_{1 \leq i,j \leq n}.$$

$s_\lambda(x)$: Schur polynomial, $h_k(x)$: complete symm. polynomial.

$\boxed{\text{Translation in the } \mathfrak{sl}_{n+1}\text{-modules}}$

$\lambda \in P^+$: dom. int. wt $\rightsquigarrow \lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$ by $\lambda_i = \sum_{k \geq i} \langle h_k, \lambda \rangle$

$\mathrm{ch}\, V(\lambda) = s_\lambda(x)$, $\mathrm{ch}\, V(k\varpi_1) = h_k(x)$ $\quad$ $(V(\lambda)$: simple $\mathfrak{sl}_{n+1}$-mod.$)$

Theorem

$$\mathrm{ch}\, V(\lambda) = \det\left(\mathrm{ch}\, V\big((\lambda_i - i + j)\varpi_1\big)\right)_{1 \leq i,j \leq n}$$

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$$\mathrm{ch}\, V(\lambda) \neq \det\Big(\mathrm{ch}\, V\big((\lambda_i - i + j)\varpi_1\big)\Big)_{1\le i,j\le n},$$

when $\mathfrak{g} \neq \mathfrak{sl}_{n+1}$ (though there may be several generalizations.)

Q. When $\mathfrak{g} \neq \mathfrak{sl}_{n+1}$, does $\det\Big(\mathrm{ch}\, V\big((\lambda_i - i + j)\varpi_1\big)\Big)_{1\le i,j\le n}$

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In type BD, we have

$$\mathrm{ch}\, L_q(\lambda) = \det\Big(\mathrm{ch}\, V\big((\lambda_i - i + j)\varpi_1\big)\Big)_{1\le i,j\le n},$$

where $L_q(\lambda)$ denotes a **minimal affinization** (a special class of f.d. simple $U_q(\mathcal{L}\mathfrak{g})$-modules explained later).

In type C, a similar formula holds:

$$\mathrm{ch}\, L_q(\lambda) = \det\Big(\sum_{0\le 2k\le \lambda_i - i + j} \mathrm{ch}\, V\big((\lambda_i - i + j - 2k)\varpi_1\big)\Big)_{1\le i,j\le n}.$$

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2. Main Theorem (JT formula for $\mathrm{ch}\, L_q(\lambda)$)
3. Proof (Combination of results proved by
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$U_q(\mathcal{L}\mathfrak{g})$: quantum loop algebra/$\mathbb{C}(q)$ $\big(q$-analog of $U(\mathcal{L}\mathfrak{g})\big)$

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<u>Fact.</u> V: an arbitrary f.d. simple $U_q(\mathcal{L}\mathfrak{g})$-module

$\rightsquigarrow {}^{\exists!}\lambda \in P^+$ s.t. $V \cong V_q(\lambda) \oplus \bigoplus_{\mu<\lambda} V_q(\mu)^{\oplus m_\mu(V)}$ as a $U_q(\mathfrak{g})$-module.

In this case, V is called an **affinization** of $V_q(\lambda)$.

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Examples of Minimal affinizations

Minimal affinizations for $\mathfrak{g} = \mathfrak{sl}_{n+1}$

When $\mathfrak{g} = \mathfrak{sl}_{n+1}$, $^{\exists}$alg. hom. $\varphi\colon U_q(\mathcal{L}\mathfrak{g}) \twoheadrightarrow U_q(\mathfrak{g})$ (evaluation map)
(q-analog of the map $\mathcal{L}\mathfrak{g} \twoheadrightarrow \mathfrak{g}\colon x \otimes f \to f(a)x$ for any $a \in \mathbb{C}^{\times}$)
$\rightsquigarrow \varphi^* V_q(\lambda)$: simple $U_q(\mathcal{L}\mathfrak{g})$-mod. $\Leftarrow$ minimal affinization of $V_q(\lambda)$

($\because \varphi^* V_q(\lambda) \cong V_q(\lambda)$ as a $U_q(\mathfrak{g})$-mod.)

Remark. If $\mathfrak{g} \neq \mathfrak{sl}_{n+1}$, evaluation map **does not** exist.
$\rightsquigarrow$ Most of minimal affinizations are reducible as a $U_q(\mathfrak{g})$-module,
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Main Theorem

In the sequel, assume that $\mathfrak{g}$ is of type $ABCD$.

Let $\lambda \in P^+$, and let $L_q(\lambda)$ be a minimal affinization of $V_q(\lambda)$.

Theorem

Assume that $\begin{cases} \langle h_n, \lambda \rangle = 0 & \text{if } \mathfrak{g}\text{: type } BC, \\ \langle h_{n-1}, \lambda \rangle = \langle h_n, \lambda \rangle = 0 & \text{if } \mathfrak{g}\text{: type } D, \end{cases}$

and set $\lambda_i := \sum_{k \geq i} \langle h_k, \lambda \rangle \in \mathbb{Z}_{\geq 0}$ for $1 \leq i \leq n$. Then we have

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$$= \begin{cases} \det\Big(\mathrm{ch}\, V\big((\lambda_i - i + j)\varpi_1\big)\Big)_{1 \leq i,j \leq n} & \mathfrak{g}\text{: } ABD \\ \det\Big(\sum_{0 \leq 2\ell \leq \lambda_i - i + j} \mathrm{ch}\, V\big((\lambda_i - i + j - 2\ell)\varpi_1\big)\Big)_{1 \leq i,j \leq n} & \mathfrak{g}\text{: } C \end{cases}$$

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Theorem

Assume that $\begin{cases} \langle h_n, \lambda \rangle = 0 & \text{if } \mathfrak{g}\text{: type } BC, \\ \langle h_{n-1}, \lambda \rangle = \langle h_n, \lambda \rangle = 0 & \text{if } \mathfrak{g}\text{: type } D, \end{cases}$

and set $\lambda_i := \sum_{k \geq i} \langle h_k, \lambda \rangle \in \mathbb{Z}_{\geq 0}$ for $1 \leq i \leq n$. Then we have

$$\mathrm{ch}\, L_q(\lambda) = \mathrm{ch}\, V(\lambda)$$

$$= \begin{cases} \det\Big(\mathrm{ch}\, V\big((\lambda_i - i + j)\varpi_1\big)\Big)_{1 \leq i,j \leq n} & \mathfrak{g}\text{: } ABD \\ \det\Big(\sum_{0 \leq 2\ell \leq \lambda_i - i + j} \mathrm{ch}\, V\big((\lambda_i - i + j - 2\ell)\varpi_1\big)\Big)_{1 \leq i,j \leq n} & \mathfrak{g}\text{: } C \end{cases}$$

Remark. In type A, this is JT formula since $\mathrm{ch}\, L_q(\lambda) = \mathrm{ch}\, V(\lambda)$.

Remark. For $k \in \mathbb{Z}_{\geq 0}$, it holds that

$$L_q(k\varpi_1) \cong_{U_q(\mathfrak{g})} \begin{cases} V_q(k\varpi_1) & \mathfrak{g}\colon ABD, \\ \bigoplus_{0 \leq 2\ell \leq k} V_q\big((k-2\ell)\varpi_1\big) & \mathfrak{g}\colon C. \end{cases}$$

Hence the theorem can be written in a uniform way as

$$\mathrm{ch}\, L_q(\lambda) = \det\Big(\mathrm{ch}\, L_q\big((\lambda_i - i + j)\varpi_1\big)\Big)_{1 \leq i,j \leq n}.$$

The multiplicity formula can be deduced from the theorem.

Corollary

$\lambda \in P^+$: as above. For every $\mu \in P^+$,

$$\Big[L_q(\lambda) : V_q(\mu)\Big]_{U_q(\mathfrak{g})} = \begin{cases} \sum_\kappa c^\lambda_{2\kappa,\mu} & \mathfrak{g}\colon BD, \\ \sum_\kappa c^\lambda_{(2\kappa)',\mu} & \mathfrak{g}\colon C. \end{cases}$$

κ: partitions, $c^\lambda_{\mu,\nu}$: Littlewood-Richardson coefficients.

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$$\mathrm{ch}\, L_q(\lambda) = \begin{cases} \det\Big(\mathrm{ch}\, V\big((\lambda_i - i + j)\varpi_1\big)\Big)_{1\le i,j\le n} & : ABD \\ \det\Big(\sum_{0\le 2\ell\le \lambda_i - i + j} \mathrm{ch}\, V\big((\lambda_i - i + j - 2\ell)\varpi_1\big)\Big)_{1\le i,j\le n} & : C \end{cases}$$

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$\boxed{\text{Graded limits}}$

$L_q(\lambda)$: $U_q(\mathcal{L}\mathfrak{g})$-mod./$\mathbb{C}(q)$ $\xrightarrow{q\to 1}$ $L_1(\lambda)$: $\mathcal{L}\mathfrak{g}$-mod./$\mathbb{C}$ (classical limit)

$\xrightarrow{\text{restrict}}$ $L_1(\lambda)$: $\mathfrak{g}[t]$-module $\quad \big(\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t] \subseteq \mathcal{L}\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]\big)$

Fact. ${}^{\exists}a \in \mathbb{C}^{\times}$ s.t. $\big(\mathfrak{g} \otimes (t+a)^N\big) L_1(\lambda) = 0 \quad (N \gg 0)$

$\rightsquigarrow$ Define an auto. τ_a on $\mathfrak{g}[t]$ by $\tau_a\big(g \otimes f(t)\big) = g \otimes f(t+a)$

$L(\lambda) := \tau_a^*\big(L_1(\lambda)\big)$: **graded limit** of $L_q(\lambda)$ ($\mathbb{Z}$-graded $\mathfrak{g}[t]$-module)

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Proposition (N)

Let $M(\lambda)$ be the $\mathfrak{g}[t]$-module generated by a vector v with relations

$$\mathfrak{n}_+[t]v = 0, \quad (h \otimes t^n)v = \delta_{0,n}\lambda(h)v \text{ for } h \in \mathfrak{h}, \quad f_i^{\lambda(h_i)+1}v = 0,$$
$$(f_\alpha \otimes t)v = 0 \text{ for } \alpha \in \Delta'_+.$$

Then the graded limit $L(\lambda)$ is isomorphic to $M(\lambda)$.

Proposition (Chari-Greenstein, 11)

$$\sum_{(\lambda,s)\in\Gamma(\mu)} (-1)^s \dim \mathrm{Hom}_{\mathfrak{g}}\Big(V(\lambda), \bigwedge^s \mathfrak{g} \otimes V(\mu)\Big) \mathrm{ch}\, M(\lambda) = \mathrm{ch}\, V(\mu),$$

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Comment on exceptional types

It would be possible to study minimal affinizations in exceptional types using their graded limits. Indeed, recently we obtain the following polyhedral multiplicity formula for minimal affinizations of type G_2:

$$L_q(k\varpi_1 + l\varpi_2) \cong_{U_q(\mathfrak{g})} \bigoplus_{(a_1,\dots,a_5)\in S(k,l)} V_q\big((k - a_1 + a_3 + a_4 - a_5)\varpi_1 + (l - a_2 - 3a_3 - 3a_4)\varpi_2\big)$$

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