Strong duality data of type A and extended T-systems

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Plan

Main Theorem

Mukhin–Young's extended T-systems are generalized to a general strong duality data of type A.

1. Mukhin–Young's extended T-systems (what we generalize)
2. Strong duality data and affine cuspidal modules (how we generalize)
3. Main Theorem
4. Proof (relations between the extended T-systems and Kashiwara crystals)

Notations

$U_q'(\mathfrak{g})$: **quantum affine algebra** with index set $[0,n]$ and $q \in \mathbb{C}^\times$ not root of 1

(assoc. alg. over $\mathbb{C}$ defined as a q-deformation of $U(\mathfrak{g})$,

where $\mathfrak{g}$: affine Lie algebra, e.g. $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}[t^{\pm 1}] \oplus \mathbb{C}K$)

$\mathscr{C}_\mathfrak{g}$: the cat. of f.d. $U_q'(\mathfrak{g})$-mod. (of type 1)

- $\mathscr{C}_\mathfrak{g}$ is a monoidal category with $\otimes$ and the trivial module 1

  $\Rightarrow K(\mathscr{C}_\mathfrak{g})$ has a ring structure (**Grothendieck ring**)
- Each $M \in \mathscr{C}_\mathfrak{g}$ has the right dual $\mathscr{D}(M)$ and the left dual $\mathscr{D}^{-1}(M)$

Theorem (Chari-Pressley, 95)

$\{\text{simples in } \mathscr{C}_\mathfrak{g}\}/_{\simeq} \overset{1:1}{\leftrightarrow} \{\boldsymbol{\pi}(u) = (\pi_1(u), \ldots, \pi_n(u)) \mid \pi_i(u) \in 1 + u\mathbb{C}[u]\}$.

Drinfeld polynomials

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Drinfeld polynomials

For $i \in [1, n]$ and $k \in \mathbb{Z}$, set

$$Y_{i,k} = Y_{i,k}(u) := (1, \dots, 1 - q^k u, \dots, 1) \in (1 + u\mathbb{C}[u])^{\times n}$$

$\rightsquigarrow$ For a sequence $\big((i_1, k_1), \dots, (i_p, k_p)\big) \in ([1, n] \times \mathbb{Z})^{\times p}$,

$$\prod_{r=1}^{p} Y_{i_r,k_r} = (\pi_1(u), \dots, \pi_p(u)), \text{ where } \pi_i(u) = \prod_{r;i_r=i} (1 - q^{k_r} u)$$

$\rightsquigarrow$ simple module $L(\prod_{r=1}^{p} Y_{i_1,k_1})$ is defined (**monomial parametrization**)

A simple module $L(Y_{i,k})$ is called a **fundamental module**.

T-systems

For general affine $\mathfrak{g}$, the **T-systems** are certain relations in $K(\mathscr{C}_\mathfrak{g})$ for the tensor product of Kirillov-Reshetikhin (KR) modules $L(\prod_{k=1}^{p} Y_{i,r+2d_i k})$:

Ex. (T-systems for untwisted, simply-laced $\mathfrak{g}$)

$$\Big[L\Big(\prod_{k=1}^{p-1} Y_{i,r+2k}\Big) \otimes L\Big(\prod_{k=2}^{p} Y_{i,r+2k}\Big)\Big]$$

$$= \Big[L\Big(\prod_{k=1}^{p} Y_{i,r+2k}\Big) \otimes L\Big(\prod_{k=2}^{p-1} Y_{i,r+2k}\Big)\Big] + \Big[\bigotimes_{c_{ij}=-1} L\Big(\prod_{k=1}^{p-1} Y_{j,r+2k+1}\Big)\Big]$$

For $\mathfrak{g}$ of type $A_n^{(1)}$ and $B_n^{(1)}$, Mukhin and Young introduced in '12 similar relations (**extended T-systems**) for **prime snake modules** (which we will recall next). These contain all T-systems of these types.

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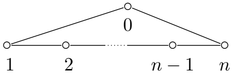
For $\mathfrak{g}$ of type $A_n^{(1)}$ and $B_n^{(1)}$, Mukhin and Young introduced in '12 similar relations (**extended T-systems**) for **prime snake modules** (which we will recall next). These contain all T-systems of these types.

Snake modules in type $A_n^{(1)}$

Assume $\mathfrak{g}$ is of type $A_n^{(1)}$:



Set $J_A := \{(i,k) \mid k \equiv i \pmod 2\} \subseteq [1,n] \times \mathbb{Z}$

$(n = 5)$

$(i \setminus k)$	$\cdots$	0	1	2	3	4	5	6	7	$\cdots$
1			◦		◦		◦		◦	
2		◦		◦		◦		◦		
3			◦		◦		◦		◦	
4		◦		◦		◦		◦		
5			◦		◦		◦		◦	

Definition

A sequence $\xi = ((i_1,k_1),\dots,(i_p,k_p)) \in J_A^p$ is a **snake**

$\overset{\text{def}}{\Leftrightarrow}$ for $1 \le {}^{\forall} r < p$, setting $(i,k) = (i_r,k_r)$ and $(i',k') = (i_{r+1},k_{r+1})$

$$|i - i'| + 2 \le k - k'$$

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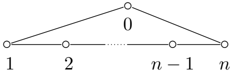

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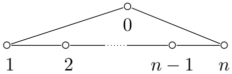

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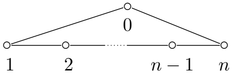

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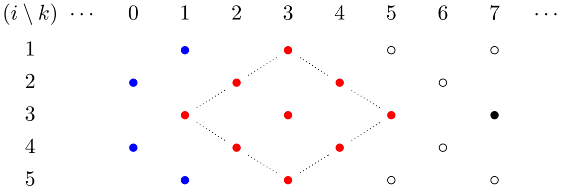

Definition

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2		$\bullet$		$\bullet$		$\circ$		$\circ$		$\circ$		$\circ$		$\circ$		
3			$\circ$		$\circ$		$\bullet$		$\circ$		$\circ$		$\circ$		$\circ$	
4		$\circ$		$\circ$		$\circ$		$\circ$		$\bullet$		$\circ$		$\circ$		
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$\boldsymbol{\xi} = \big((i_1, k_1), \dots, (i_p, k_p)\big)$: snake $\Rightarrow$ $L(\boldsymbol{\xi}) = L(\prod_{r=1}^{p} Y_{i_r, k_r})$: **snake module**

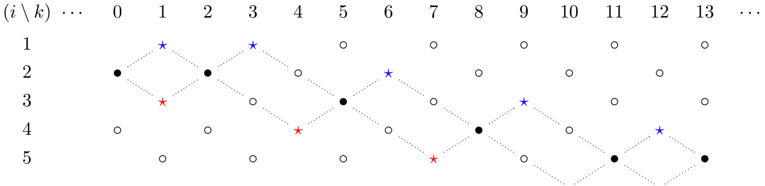
prime snake $\xi \rightsquigarrow$ two **neighboring snakes** ξ_{H} $(\star)$, ξ_{L} $(\star)$

Theorem (MY12)

- ξ: prime $\Leftrightarrow$ $L(\xi)$: prime (i.e. $L(\xi) \simeq M \otimes N \Rightarrow M \simeq \mathbf{1}$ or $N \simeq \mathbf{1}$)
- $[L(\prod_{r=1}^{p-1} Y_{i_r,k_r}) \otimes L(\prod_{r=2}^{p} Y_{i_r,k_r})] = [L(\xi) \otimes L(\prod_{r=2}^{p-1} Y_{i_r,k_r})] + [L(\xi_{\mathrm{H}}) \otimes L(\xi_{\mathrm{L}})]$

  simple

Rem. KR module $\Leftrightarrow$ straight snake $\bullet \quad \bullet \quad \bullet \quad \cdots$



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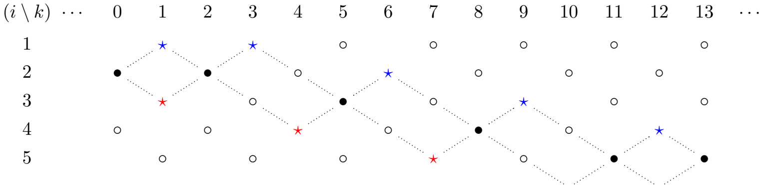

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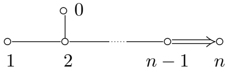
Extended T-systems in type $B_n^{(1)}$



$$J_B = \{(i,k) \mid k \equiv \delta_{in} \pmod 2\} \subseteq [1,n] \times \mathbb{Z}$$

(n=3)

$(i \setminus k)$ $\cdots$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	$\cdots$
1	○				○				○				○				○				○				○	
2			○				○				○				○				○				○			
3		○		○		○		○		○		○		○		○		○		○		○		○		
2	○				○				○				○				○				○				○	
1			○				○				○				○				○				○			

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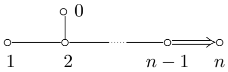

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	1		○				○				○				○				○				○				○	
(n=3)	2				●				●				○				○				○				○			
	3			○		○		○		○		○		●		○		○		○		○		●		●		
	2		○				○				○				○				●				○				○	
	1				○				○				○				○				○				○			

$\boldsymbol{\xi} = ((i_1,k_1),\ldots,(i_p,k_p))$: (prime) snake

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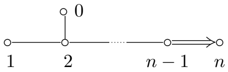

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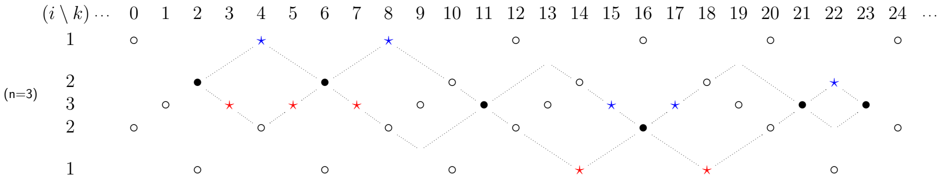

$\boldsymbol{\xi} = ((i_1,k_1),\ldots,(i_p,k_p))$: prime snake $\rightsquigarrow$ neighboring snakes $\boldsymbol{\xi}_{\mathrm{H}}, \boldsymbol{\xi}_{\mathrm{L}}$

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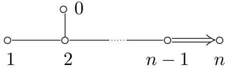
Extended T-systems in type $B_n^{(1)}$



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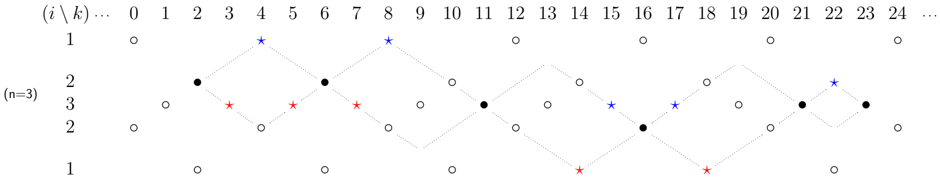

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Extended T-systems

$$[L(\prod_{r=1}^{p-1} Y_{i_r,k_r}) \otimes L(\prod_{r=2}^{p} Y_{i_r,k_r})] = [L(\boldsymbol{\xi}) \otimes L(\prod_{r=2}^{p-1} Y_{i_r,k_r})] + [L(\boldsymbol{\xi}_{\mathrm{H}}) \otimes L(\boldsymbol{\xi}_{\mathrm{L}})]$$

This was proved by showing that the q-**characters** of both sides coincide.

Questions

1. Are there other families of simple modules satisfying these relations?
2. Why snake modules satisfy these relations? In other words, where the prime snake condition for highest monomials come from?

Relations with cluster algebras

Theorem (Hernandez–Leclerc '10, Kashiwara–Kim–Oh–Park '22)

A subcat. $\mathscr{C}_{\mathfrak{g}}^{-} \subseteq \mathscr{C}_{\mathfrak{g}}$ is a monoidal categorification of a cluster alg. $\mathscr{A}$, i.e., we have

1. $\psi \colon \mathscr{A} \xrightarrow{\sim} K(\mathscr{C}_{\mathfrak{g}}^{-})$.
2. ψ(cluster var.) $\subsetneq$ (prime real simple mod. in $\mathscr{C}_{\mathfrak{g}}^{-}$). ($M$: **real** $\overset{\text{def}}{\Leftrightarrow}$ $M \otimes M$: simple)

$\Rightarrow xy = \prod_i z_i + \prod_i w_i$ (mutation in $\mathscr{A}$)

$\rightsquigarrow [\psi(x) \otimes \psi(y)] = \left[\bigotimes_i \psi(z_i)\right] + \left[\bigotimes_i \psi(w_i)\right]$ (relations in $K(\mathscr{C}_{\mathfrak{g}}^{-})$).

The initial seed corresponds to KR modules, and T-systems are mutations.

It is strongly expected that extended T-systems also come from mutations (though this has not been proved so far). In fact, it is known that all prime snake modules correspond to cluster variables [Duan–Li–Luo, 19].

In this view, extended T-systems are mutations **having distinguished combinatorial description**.

Main Theorem

Mukhin–Young's extended T-systems are generalized to a general strong duality data of type A.

1. Mukhin–Young's extended T-systems (what we generalize)
2. Strong duality data and affine cuspidal modules [KKOP]
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invariants $\mathfrak{d}(M, N)$

For simple modules $M, N \in \mathscr{C}_{\mathfrak{g}}$, there is an isomorphism

$$R^{\mathrm{norm}}_{M,N} \colon \mathbb{C}(z) \otimes_{\mathbb{C}[z^{\pm 1}]} (M \otimes N[z^{\pm 1}]) \xrightarrow{\sim} \mathbb{C}(z) \otimes_{\mathbb{C}[z^{\pm 1}]} (N[z^{\pm 1}] \otimes M)$$

(normalized R-matrix)

Rem. An isom. $M \otimes N \xrightarrow{\sim} N \otimes M$ does not necessarily exist in $\mathscr{C}_{\mathfrak{g}}$.

Definition

For simple modules M, N of $\mathscr{C}_{\mathfrak{g}}$, define $\mathfrak{d}(M, N) \in \mathbb{Z}_{\geq 0}$ by

$\mathfrak{d}(M, N) :=$

(deg. of the pole of $R^{\mathrm{norm}}_{M,N}$ at $z = 1$) + (deg. of the pole of $R^{\mathrm{norm}}_{N,M}$ at $z = 1$)

Intuitively, $\mathfrak{d}(M, N)$ measures how far from the existence of an isomorphism $M \otimes N \xrightarrow{\sim} N \otimes M$.

invariants $\mathfrak{d}(M, N)$

For simple modules $M, N \in \mathscr{C}_{\mathfrak{g}}$, there is an isomorphism

$$R^{\mathrm{norm}}_{M,N} \colon \mathbb{C}(z) \otimes_{\mathbb{C}[z^{\pm 1}]} (M \otimes N[z^{\pm 1}]) \xrightarrow{\sim} \mathbb{C}(z) \otimes_{\mathbb{C}[z^{\pm 1}]} (N[z^{\pm 1}] \otimes M)$$

(normalized R-matrix)

Rem. An isom. $M \otimes N \xrightarrow{\sim} N \otimes M$ does not necessarily exist in $\mathscr{C}_{\mathfrak{g}}$.

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strong duality data

Fix Cartan matrix $C = (c_{ij})_{i,j \in I}$ of finite ADE type (irrelevant to the type of $\mathfrak{g}$)

Definition

A family of real simple modules $\mathcal{D} = \{L_i\}_{i \in I} \subseteq \mathscr{C}_{\mathfrak{g}}$ is called a **strong duality datum** (associated with C) if

1. $\mathfrak{d}(L_i, \mathscr{D}^k L_i) = \delta_{k,0} \quad (^\forall i \in I, ^\forall k \in \mathbb{Z})$,
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Proposition (KKOP)

$\mathcal{D} = \{L_i\}_{i \in I} \subseteq \mathscr{C}_{\mathfrak{g}}$: a strong duality datum associated with C

$\Rightarrow$ $^{\exists !}\mathbb{Z}$-alg. hom. $\Phi_{\mathcal{D}} \colon U_q^-(\mathfrak{g}_C)^\vee_{\mathbb{Z}} \to K(\mathscr{C}_{\mathfrak{g}})$ s.t. $\Phi_{\mathcal{D}}(f_i) = [L_i]$ $(i \in I)$, $\Phi_{\mathcal{D}}(q) = 1$.

Moreover, $\Phi_{\mathcal{D}}$ induces an inj. map from the **upper global basis** $\mathbf{B}^{\mathrm{up}} \subseteq U_q^-(\mathfrak{g}_C)^\vee_{\mathbb{Z}}$ to the isom. classes of simple modules in $\mathscr{C}_{\mathfrak{g}}$.

Rem. $\Phi_{\mathcal{D}}$ is defined by the composition of the following two hom.:

- $U_q^-(\mathfrak{g}_C)^\vee_{\mathbb{Z}} \xrightarrow{\sim} K(R^C\text{-gmod})$ (R^C: **quiver Hecke algebra**) [Khovanov–Lauda, Rouquier]
- $K(R^C\text{-gmod}) \to K(\mathscr{C}_{\mathfrak{g}})$, which is induced from the **quantum affine Schur–Weyl duality functor** $R^C\text{-gmod} \to \mathscr{C}_{\mathfrak{g}}$.

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affine cuspidal modules

$\mathcal{D}$: strong duality datum associated with C

$$\rightsquigarrow \Phi_{\mathcal{D}} \colon \begin{array}{ccc} U_q^-(\mathfrak{g}_C)^\vee_{\mathbb{Z}} & \to & K(\mathscr{C}_{\mathfrak{g}}) \\ \cup & & \cup \\ \mathbf{B}^{\mathrm{up}} & & \{\text{simple mod.}\}/_{\cong} \end{array}$$

Fix a reduced word $\boldsymbol{i} = (i_1, \ldots, i_N)$ of the longest el. w_0 of W_C. For $1 \leq j \leq N$,

$U_q^-(\mathfrak{g}_C)^\vee \ni f_{\beta_j} := T_{i_1} \cdots T_{i_{j-1}}(f_{i_j}) \overset{\text{normalize}}{\longrightarrow} f^\vee_{\beta_j} \in \mathbf{B}^{\mathrm{up}}$: **dual root vector**

(T_i: Lusztig's braid group action)

Rem. $f^\vee_{\beta_j}$ depends on the choice of $\boldsymbol{i}$.

Definition

Define the **affine cuspidal modules** $\{S_j = S_j^{\mathcal{D},\boldsymbol{i}} \mid j \in \mathbb{Z}\} \subseteq \mathscr{C}_{\mathfrak{g}}$ as follows:

(i) if $1 \leq j \leq N$, S_j is the image of $f^\vee_{\beta_j}$ under $\Phi_{\mathcal{D}}$, and

(ii) set $S_{j \pm N} = \mathscr{D}^{\mp 1} S_j$ for all $j \in \mathbb{Z}$.

affine cuspidal modules

$\mathcal{D}$: strong duality datum associated with C

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Examples

(1) Assume $\mathfrak{g} = \widehat{\mathfrak{sl}}_{n+1}$ (type $A_n^{(1)}$).

$\mathcal{D}^A := \{L(Y_{1,-2j+1}) \mid 1 \le j \le n\} \subseteq \mathscr{C}_{\widehat{\mathfrak{sl}}_{n+1}}$: SDD of type A_n

$\boldsymbol{i}^A := (1, \ldots, n/1, \ldots, n-1/\ldots/1, 2/1)$

$\rightsquigarrow$ …, $S_1 = \Phi_{\mathcal{D}}(f_1) = L(Y_{1,-1})$, $S_2 = \Phi_{\mathcal{D}}(f^{\vee}_{\alpha_1+\alpha_2}) = L(Y_{2,-2})$, …,

$S_{N+1} = \mathscr{D}^{-1}(S_1) = L(Y_{n,-n-2}), \ldots \rightsquigarrow \{S_j^{\mathcal{D}^A, \boldsymbol{i}^A} \mid j \in \mathbb{Z}\} = \{L(Y_{i,k}) \mid k \equiv i (\mathrm{mod}\ 2)\}$

(2) Assume $\mathfrak{g} = \widehat{\mathfrak{so}}_n$ with odd n (type $B_{n_0}^{(1)}$ with $n = 2n_0 - 1$).

$$L_j = \begin{cases} L(Y_{1,-4j+2})\ (j < n_0), & L(Y_{n_0,-3n+2})\ (j = n_0) \\ L(Y_{n_0,-n-2})\ (j = n_0+1), & L(Y_{1,-4j+6})\ (j > n_0) \end{cases}$$

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Rem. For general $\mathfrak{g}$, fund. mod. $L(Y_{i,k})$ are ACM assoc. with suitable $\mathcal{D}, \boldsymbol{i}$.

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Main Theorem

Mukhin–Young's extended T-systems are generalized to a general strong duality data of type A.

1. Mukhin–Young's extended T-systems (what we generalize)
2. Strong duality data and affine cuspidal modules [KKOP]
3. Main Theorem
4. Proof (relations between the extended T-systems and Kashiwara crystals)

Prior work for T-systems by [KKOP]

$\mathcal{D} \subseteq \mathscr{C}_{\mathfrak{g}}$: SDD of arbitrary type, $\boldsymbol{i}$: arbitrary reduced word of the longest el. w_0,

$\rightsquigarrow \{S_j = S_j^{\mathcal{D},\boldsymbol{i}} \mid j \in \mathbb{Z}\}$: the associated affine cuspidal modules

A family of simple modules $\{M_i[a,b] \mid i \in I,\ a < b\}$ was defined, where

$M_i[a,b] := \mathrm{hd}(S_{r_1} \otimes \cdots \otimes S_{r_p})$ with a suitable seq. $r_1 < r_2 < \cdots < r_p$ of integers.

Fact In the case where $\{S_j\}$ are fundamental modules,

$$M_i[a,b] = \mathrm{hd}\big(L(Y_{i,r+2}) \otimes \cdots \otimes L(Y_{i,r+2k})\big) \cong L\Big(\prod_{k=1}^{p} Y_{i,r+2k}\Big)\text{: KR modules.}$$

Theorem (KKOP)

$$0 \to \bigotimes_{j;c_{ij}=-1} M_j[a+1,b-1] \to M_i[a,b-1] \otimes M_i[a+1,b] \to M_i[a,b] \otimes M_i[a+1,b-1] \to 0$$

- The first and the third terms are both simple.

$\rightsquigarrow$ T-systems as the special cases of KR modules

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Main Theorem

Setting $\mathcal{D} \in \mathscr{C}_{\mathfrak{g}}$: a strong duality datum of type A_n, $X \in \{A, B\}$

$\boldsymbol{i}^X$: the reduced word defined in the previous slide

(i.e. $\boldsymbol{i}^A = (1, \dots, n/\dots/1, 2/1)$, $\boldsymbol{i}^B = (1, \dots, n/n_0/\dots/1, \dots, n_0/\dots/12/1)$)

$\rightsquigarrow$ affine cuspidal modules $S_j^{\mathcal{D}, \boldsymbol{i}^X}$ $(j \in \mathbb{Z})$

Set $J_A := \{(i,k) \mid k \equiv i \pmod 2\}$, $J_B := \{(i,k) \mid k \equiv \delta_{i,n_0} \pmod 2\}$, and define

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where $j \in \mathbb{Z}$ is s.t. $S_j^{\mathcal{D}^X, \boldsymbol{i}^X} = L(Y_{i,k})$.

Here $\mathcal{D}^X$ is the special SDD s.t. $\{S_j^{\mathcal{D}^X, \boldsymbol{i}^X} \mid j \in \mathbb{Z}\} = \{L(Y_{i,k}) \mid (i,k) \in J_X\}$

Definition (Snake module associated with $(\mathcal{D}, \boldsymbol{i}^X)$)

$\mathbb{S}^X(\xi) := \mathrm{hd}(S_{i_1,k_1}^X \otimes \cdots \otimes S_{i_p,k_p}^X)$ for a snake $\xi = ((i_1,k_1), \dots, (i_p.k_p)) \in J_X^p$.

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Theorem (N)

For $X \in \{A,B\}$ and a prime snake $\boldsymbol{\xi} \in J^p_X$,

$$0 \to \mathbb{S}^X(\boldsymbol{\xi}_{\mathrm{H}}) \otimes \mathbb{S}^X(\boldsymbol{\xi}_{\mathrm{L}}) \to \mathbb{S}^X(\boldsymbol{\xi}_{[1,p-1]}) \otimes \mathbb{S}^X(\boldsymbol{\xi}_{[2,p]}) \to \mathbb{S}^X(\boldsymbol{\xi}) \otimes \mathbb{S}^X(\boldsymbol{\xi}_{[2,p-1]}) \to 0$$

- The first and the third terms are simple.

$\rightsquigarrow$ $[L(\boldsymbol{\xi}_{[1,p-1]}) \otimes L(\boldsymbol{\xi}_{[2,p]})] = [L(\boldsymbol{\xi}) \otimes L(\boldsymbol{\xi}_{[2,p-1]})] + [L(\boldsymbol{\xi}_{\mathrm{H}}) \otimes L(\boldsymbol{\xi}_{\mathrm{L}})]$ when $\mathcal{D} = \mathcal{D}^X$.

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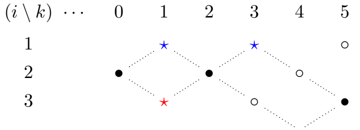
- The first and the third terms are simple.

$\rightsquigarrow$ $\big[L(\boldsymbol{\xi}_{[1,p-1]}) \otimes L(\boldsymbol{\xi}_{[2,p]})\big] = \big[L(\boldsymbol{\xi}) \otimes L(\boldsymbol{\xi}_{[2,p-1]})\big] + \big[L(\boldsymbol{\xi}_{\mathrm{H}}) \otimes L(\boldsymbol{\xi}_{\mathrm{L}})\big]$ when $\mathcal{D} = \mathcal{D}^X$.

<u>Ex.</u> type A_3, $\boldsymbol{i}^A = (1,2,3,1,2,1)$



$\rightsquigarrow$

$$0 \to \mathbb{S}^A\big((1,3),(1,1)\big) \otimes \mathbb{S}^A\big((3,1)\big)$$
$$\to \mathbb{S}^A\big((3,5),(2,2)\big) \otimes \mathbb{S}^A\big((2,2),(2,0)\big)$$
$$\to \mathbb{S}^A\big((3,5),(2,2),(2,0)\big) \otimes \mathbb{S}^A\big((2,2)\big) \to 0$$

(i) $L_i = L(Y_{1,-2i+1}) \rightsquigarrow S^A_{i,k} = L(Y_{i,k}) \rightsquigarrow \mathbb{S}^A\big((i_1,k_1),\cdots,(i_p,k_p)\big) = L(\prod_{r=1}^{p} Y_{i_r,k_r})$

$$L(Y_{1,3}Y_{1,1}) \otimes L(Y_{3,1}) \to L(Y_{3,5}Y_{2,2}) \otimes Y(Y_{2,2}Y_{2,0}) \to L(Y_{3,5}Y_{2,2}Y_{2,0}) \otimes L(Y_{2,2})$$

(ii) $L_i = L(Y_{1,2i-7}) \rightsquigarrow S^A_{3,5} = L(Y_{1,7}Y_{1,5}Y_{1,3})$, $S^A_{2,2} = L(Y_{3,1}Y_{3,-1})$,

$S^A_{2,0} = L(Y_{3,3}Y_{3,1})$, $S^A_{1,3} = L(Y_{1,7})$, …, etc.

$$\rightsquigarrow \mathbb{S}^A\big((3,5),(2,2),(2,0)\big) = \mathrm{hd}\big(L(Y_{1,7}Y_{1,5}Y_{1,3}) \otimes L(Y_{3,1}Y_{3,-1}) \otimes L(Y_{3,3}Y_{3,1})\big)$$
$$= L(Y_{1,7}Y_{1,5}Y_{1,3}Y_{2,2}Y_{2,0}), \text{ etc.}$$

$$L(Y_{1,7}Y_{3,3}Y_{3,1}Y_{3,-1}) \otimes L(Y_{1,-3}) \to L(Y_{1,7}Y_{1,5}Y_{1,3}Y_{3,1}Y_{3,-1}) \otimes L(Y_{2,2}Y_{2,0})$$
$$\to L(Y_{1,7}Y_{1,5}Y_{1,3}Y_{2,2}Y_{2,0}) \otimes L(Y_{3,1}Y_{3-1})$$

Main Theorem

Mukhin–Young's extended T-systems are generalized to a general strong duality data of type A.

1. Mukhin–Young's extended T-systems (what we generalize)
2. Strong duality data and affine cuspidal modules [KKOP]
3. Main Theorem
4. Proof (relations between the extended T-systems and Kashiwara crystals)

Key Lemma

$\mathcal{D}, \boldsymbol{i}$: arbitrary $\rightsquigarrow \{S_j = S_j^{\mathcal{D},\boldsymbol{i}} \mid j \in \mathbb{Z}\}$: affine cuspidal modules

Lemma (KKOP, N)

For a sequence $\boldsymbol{k} = (k_1 < k_2 < \cdots < k_p) \in \mathbb{Z}^p$, denote by

$$\mathbb{S}_{\boldsymbol{k}}[s,t] = \mathrm{hd}(S_{k_s} \otimes S_{k_{s+1}} \otimes \cdots \otimes S_{k_t}) \quad (1 \le s \le t \le p).$$

Assume that

(i) $\mathfrak{d}\big(S_{k_s}, \mathbb{S}_{\boldsymbol{k}}[s+1,t]\big) = 1$ for all $1 \le s < t \le p$,

(ii) $\mathfrak{d}\big(\mathbb{S}_{\boldsymbol{k}}[s,t-1], S_{k_t}\big) = 1$ for all $1 \le s < t \le p$.

Then we have

$$0 \to \mathrm{hd}\Big(\bigotimes_{r=1}^{p-1} \mathrm{hd}(S_{k_{r+1}} \otimes S_{k_r})\Big) \to \mathbb{S}_{\boldsymbol{k}}[1,p-1] \otimes \mathbb{S}_{\boldsymbol{k}}[2,p] \to \mathbb{S}_{\boldsymbol{k}}[1,p] \otimes \mathbb{S}_{\boldsymbol{k}}[2,p-1] \to 0.$$

Moreover, the first and the third terms are both simple.

Q. How to calculate the values of $\mathfrak{d}$?

$$\Phi_{\mathcal{D}}\colon \quad U_q^-(\mathfrak{g}_C)^\vee_{\mathbb{Z}} \quad \to \quad K(\mathscr{C}_{\mathfrak{g}})$$

$$\cup \qquad\qquad\qquad \cup$$

$$\{(f^\vee_{\beta_1})^{a_1}\cdots(f^\vee_{\beta_N})^{a_N} \mid \boldsymbol{a}=(a_1,\dots,a_N)\in\mathbb{Z}^N_{\geq 0}\} \qquad \{S_1^{\otimes a_1}\otimes\cdots\otimes S_N^{\otimes a_N} \mid \boldsymbol{a}\in\mathbb{Z}^N_{\geq 0}\}$$

dual PBW basis (depending on $\boldsymbol{i}$) $\qquad \updownarrow$

$$\mathbf{B}^{\mathrm{up}} = \{b^{\boldsymbol{i}}(\boldsymbol{a}) \mid \boldsymbol{a}\in\mathbb{Z}^N_{\geq 0}\} \qquad \{\mathrm{hd}(S_1^{\otimes a_1}\otimes\cdots\otimes S_N^{\otimes a_N}) \mid \boldsymbol{a}\in\mathbb{Z}^N_{\geq 0}\}$$

Rem. $\mathbf{B}^{\mathrm{up}}$ has a Kashiwara (bi-)crystal structure.

Lemma (KKOP)

1. $\mathfrak{d}(\mathscr{D}L_i, \mathrm{hd}(S_1^{\otimes a_1}\otimes\cdots\otimes S_N^{\otimes a_N})) = \varepsilon_i(b^{\boldsymbol{i}}(\boldsymbol{a}))$,
2. $\mathfrak{d}(\mathrm{hd}(S_1^{\otimes a_1}\otimes\cdots\otimes S_N^{\otimes a_N}), \mathscr{D}^{-1}L_i) = \varepsilon_i^*(b^{\boldsymbol{i}}(\boldsymbol{a}))$,

where $\varepsilon_i(b) = \max\{r\in\mathbb{Z}_{\geq 0} \mid \tilde{e}_i^r(b)\neq 0\}$, and ε_i^* is defined similarly for $\tilde{e}_i^*$.

$$\Phi_{\mathcal{D}}: \quad U_q^-(\mathfrak{g}_C)^\vee_{\mathbb{Z}} \quad \to \quad K(\mathscr{C}_{\mathfrak{g}})$$

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Lemma (KKOP)

1. $\mathfrak{d}\big(\mathscr{D}L_i, \mathrm{hd}(S_1^{\otimes a_1}\otimes\cdots\otimes S_N^{\otimes a_N})\big) = \varepsilon_i\big(b^{\boldsymbol{i}}(\boldsymbol{a})\big)$,
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where $\varepsilon_i(b) = \max\{r\in\mathbb{Z}_{\geq 0} \mid \tilde{e}_i^r(b)\neq 0\}$, and ε_i^* is defined similarly for $\tilde{e}_i^*$.

We return to the setting of our main theorem ($\mathcal{D}$: type A, $\boldsymbol{i} \in \{\boldsymbol{i}^A, \boldsymbol{i}^B\}$).

(The case of $\boldsymbol{i} = \boldsymbol{i}^A$)

Lemma

For $\mathbb{S} := \mathrm{hd}(S_k^{\otimes a_k} \otimes S_{k+1}^{\otimes a_{k+1}} \otimes \cdots \otimes S_\ell^{\otimes a_\ell})$, the following are equivalent:

(i) $\mathbb{S}$ is a prime snake module,

(ii) for all $k \le s < t \le \ell$, we have $\varepsilon_{i_s}(b^{\boldsymbol{i}'}(\boldsymbol{a}')) = \varepsilon^*_{i_t}(b^{\boldsymbol{i}''}(\boldsymbol{a}'')) = 1$,
where i_r ($r \in \mathbb{Z}$) is determined from $\boldsymbol{i} = (i_1, \ldots, i_N)$ by $i_{r+N} = i_r^*$,
$\boldsymbol{i}' = (i_{s+1}, \ldots, i_{s+N})$ and $\boldsymbol{a}' = (a_{s+1}, \ldots, a_{s+N})$. $\boldsymbol{i}''$ and $\boldsymbol{a}''$ are defined similarly,

(iii) for all $k \le s < t \le \ell$, we have

$$\mathfrak{d}\big(S_s, \mathrm{hd}(S_{s+1}^{\otimes a_{s+1}} \otimes \cdots \otimes S_t^{\otimes a_t})\big) = \mathfrak{d}\big(\mathrm{hd}(S_s^{\otimes a_s} \otimes \cdots \otimes S_{t-1}^{\otimes a_{t-1}}), S_t\big) = 1.$$

$\because$ (i) $\Leftrightarrow$ (ii): follows from the **Reineke's algorithm**, which gives a useful combinatorial algorithm for $\varepsilon_i(b^{\boldsymbol{i}}(\boldsymbol{a}))$ and $\varepsilon^*_i(b^{\boldsymbol{i}}(\boldsymbol{a}))$, when $\boldsymbol{i}$ is "adapted".

(ii) $\Leftrightarrow$ (iii): follows from the previous lemma.

(The case of $\boldsymbol{i} = \boldsymbol{i}^B$) not adapted

<u>Fact</u> $\boldsymbol{i} = (\dots, i_{k-1}, i_k, i_{k+1}, \dots) \overset{\text{3-move}}{\rightsquigarrow} \boldsymbol{i}' = (\dots, i'_{k-1}, i'_k, i'_{k+1}, \dots)$,

$\Rightarrow b^{\boldsymbol{i}}(\boldsymbol{a}) = b^{\boldsymbol{i}'}(\boldsymbol{a}')$ with $a'_{k-1} = a_k + a_{k+1} - \min(a_{k-1}, a_{k+1})$,

$a'_k = \min(a_{k-1}, a_{k+1}), a'_{k+1} = a_{k-1} + a_k - \min(a_{k-1}, a_{k+1})$ [Lusztig]

$\rightsquigarrow$ for $\boldsymbol{a} \in \mathbb{Z}^N_{\geq 0}$, we can calculate (in principal) $\boldsymbol{a}'$ s.t. $b^{\boldsymbol{i}^B}(\boldsymbol{a}) = b^{\boldsymbol{i}^A}(\boldsymbol{a}')$.

$\rightsquigarrow$ reduced to the previous case of $\boldsymbol{i}^A$.

<u>Future work</u> Generalize to other types of strong duality data.

One obstacle is that in other types, the Reineke's algorithm cannot be applied (in full generality).

Thank you for your concentration!

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