Equivalence via quantum affine Schur–Weyl duality

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Summary of today's result

$$\begin{pmatrix}\text{f.d. mod. over a} \\ \text{quantum affine alg. } U_q'(\mathfrak{g})\end{pmatrix}$$

$$\bigcup$$

$$\begin{pmatrix}\text{f.d. mod. over} \\ \text{quiver Hecke algebras } R(\beta)\end{pmatrix} \xrightarrow[\text{by Kang–Kashiwara–Kim}]{\mathcal{F}\text{: gen. Q-affine SW duality functor}} \begin{pmatrix}\text{A generalization of} \\ \text{Hernandez–Leclerc's} \\ \text{subcategory } \mathcal{C}_{\mathbf{Q}}\end{pmatrix}$$

Theorem ([N])

In a general affine type, $\mathcal{F}$ gives an equivalence of two monoidal categories.

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Notation $A\text{-Mod}$: cat. of f.g. A-modules

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Quiver Hecke algebras $R(\beta)$

Khovanov–Lauda [KL09] and Rouquier [Rou08] defined independently.

Given a Kac-Moody g (or its Cartan matrix A)

$\rightsquigarrow$ $R(\beta)$: **quiver Hecke algebras** (family of algebras, $\beta \in Q^+ = \sum_i \mathbb{Z}_{\geq 0}\alpha_i$)

$\circ$ $R(\beta)$ are $\mathbb{Z}$-graded algebras,

$\circ$ $M \in R(\beta)\text{-gmod}$, $M' \in R(\beta')\text{-gmod}$,

$\rightsquigarrow$ $M \circ M' \in R(\beta + \beta')\text{-gmod}$: **convolution product**

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Theorem ([KL09],[Rou08])

$\bigoplus_\beta K(R(\beta)\text{-gmod}) \cong U_{\mathbf{A}}^-(\mathfrak{g})^\vee$: int. form of the dual of the half of $U_q(\mathfrak{g})$ (as $\mathbb{Z}[q^{\pm 1}]$-algebra)

Theorem ([Varagnolo-Vasserot, 11], [Rouquier, 12])

$\mathfrak{g}$: symmetric $\Rightarrow$ the isom. sends simples to the upper global basis.

$$\bigoplus_{\beta} K(R(\beta)\text{-gmod}) \cong U_{\mathbf{A}}^{-}(\mathrm{g})^{\vee}$$

$$\cup \qquad\qquad \cup$$

$$\{\text{simples}\} \to \{\text{upper global basis}\}$$

By specializing at $q = 1$, we obtain the following.

Corollary

If g is a simple Lie algebra of type ADE,

(i) $\bigoplus_{\beta} \mathbb{C} \otimes_{\mathbb{Z}} K(R(\beta)\text{-mod}^0) \cong \mathbb{C}[N]$,

where $R(\beta)\text{-mod}^0$: cat. of f.d. mod. on which x_k's act nilpotently

(obtained from graded ones by forgetting the gradings)

$\mathbb{C}[N]$: coordinate ring of the unipotent group associated with g.

(ii) This isom. sends simples to (the specialization of) upper global basis.

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There is another algebra categorifying the same things!

Hernandez–Leclerc's subcategory

[Hernandez–Leclerc, 15]

$\mathfrak{g}$: simple Lie algebra of type ADE, R^+: positive roots of $\mathfrak{g}$

$\hat{\mathfrak{g}}$: untwisted affine Lie algebra associated with $\mathfrak{g}$,

$\mathcal{C}_{\hat{\mathfrak{g}}}$: cat. of f.d. $U_q'(\hat{\mathfrak{g}})$-modules ($U_q'(\widehat{\mathfrak{g}})$: quantum group of $\widehat{\mathfrak{g}}$)

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They defined a map $Q\colon R^+ \ni \alpha \mapsto V^\alpha \in \mathcal{C}_{\hat{g}}$: simple (fundamental) modules (using the theory of Auslander-Reiten quiver)

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Theorem

$\mathbb{C} \otimes_{\mathbb{Z}} K(\mathcal{C}_Q) \cong \mathbb{C}[N]$ as a $\mathbb{C}$-algebra, and this sends simples to (the specialization of) upper global basis.

$$\bigoplus_{\beta} \mathbb{C} \otimes_{\mathbb{Z}} K(R(\beta)\text{-mod}^0) \cong \quad \mathbb{C}[N] \quad \cong \ \mathbb{C} \otimes_{\mathbb{Z}} K(\mathcal{C}_Q)$$

$$\text{(simples)} \qquad \leftrightarrow \text{(gl. basis)} \leftrightarrow \quad \text{(simples)}$$

Q. Is there a functor between $R(\beta)\text{-mod}^0$ and $\mathcal{C}_Q$ inducing this isomorphism?

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<u>Type A</u> [Chari–Pressley, Cherednik, Ginzburg–Varagnolo–Vasserot]

$R(\beta)\text{-mod}^0 \fallingdotseq H_q^{\mathrm{aff}}(d)\text{-mod}$ (affine Hecke algebra)

$\mathbb{V}^{\otimes d}$: $(U_q'(\widehat{\mathfrak{sl}}_n), H_q^{\mathrm{aff}}(d))$-bimodule

$\Rightarrow H_q^{\mathrm{aff}}(d)\text{-mod} \ni M \mapsto \mathbb{V}^{\otimes d} \otimes_{H_q^{\mathrm{aff}}(d)} M \in \mathcal{C}_{\widehat{\mathfrak{sl}}_n}$

(**quantum affine Schur–Weyl duality functor**)

Kang–Kashiwara–Kim's construction of functors

[KKK18]: construction of functors in general setting

[KKK15]: application of the results in [KKK18] to HL subcategories
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$U_q'(\mathfrak{g})$: quantum affine algebra of a general affine type

Given a family of real simple modules $\{V_i\}_{i \in J} \in \mathcal{C}_\mathfrak{g}$

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$U_q'(\mathfrak{g})$: quantum affine algebra of a general affine type

Given a family of real simple modules $\{V_i\}_{i\in J} \in \mathcal{C}_\mathfrak{g}$

$\rightsquigarrow$ define a Cartan matrix $A = (a_{ij})_{i,j\in J}$ by

$$a_{ij} = \begin{cases} 2 & (i = j), \\ -b_{ij} - b_{ji} & (i \neq j), \end{cases}$$ where

$b_{ij} = \left(\text{deg. of pole of } V_i \otimes (V_j[z^{\pm 1}]) \overset{R^{\mathrm{norm}}}{\to} (V_j(z)) \otimes V_i \text{ at } z = 1\right)$.

$\rightsquigarrow$ $\{R(\beta)\}_{\beta\in Q^+}$: quiver Hecke algebras assoc. with A

Then we construct a $(U_q'(\mathfrak{g}), R(\beta))$-bimodule as follows.

V_i $(i \in J) \rightsquigarrow \widehat{V}_i = V_i[\![w]\!]$: a completed affinization $\quad$ ($U_q'(\mathfrak{g})$-module)

For $\beta \in Q^+$, $\widehat{V}^{\otimes\beta} = \displaystyle\bigoplus_{\alpha_{i_1}+\cdots+\alpha_{i_p}=\beta} \widehat{V}_{i_1} \hat{\otimes} \cdots \hat{\otimes} \widehat{V}_{i_p}$.

$U_q'(\mathfrak{g}) \curvearrowright \widehat{V}^{\otimes\beta} \curvearrowleft R(\beta)$ defined using R-matrices

$\rightsquigarrow \mathcal{F}_\beta \colon R(\beta)\text{-mod}^0 \to \mathcal{C}_\mathfrak{g}, \quad M \mapsto \widehat{V}^{\otimes\beta} \otimes_{R(\beta)} M$

$\mathcal{F} = \bigoplus_\beta \mathcal{F}_\beta \colon \bigoplus_\beta R(\beta)\text{-mod}^0 \to \mathcal{C}_\mathfrak{g}$: **gene'd Q-aff. SW duality functor**

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Theorem ([KKK18])

(i) $\mathcal{F}$ is monoidal ($\mathcal{F}(M \circ M') \cong \mathcal{F}(M) \otimes \mathcal{F}(M')$, etc.).

(ii) If $\{R(\beta)\}$ are of type ADE, $\mathcal{F}$ is exact.

In [KKK15], a functor $\mathcal{F}\colon \bigoplus_{\beta \in Q^+} R(\beta)\text{-mod}^0 \to \mathcal{C}_Q$ in untwisted ADE types ($\mathfrak{g} = \widehat{\mathrm{g}}$) was constructed using the results of [KKK18], which gives an answer to the previous question.

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recall In the construction of $\mathcal{C}_Q$, defined a map $R^+ \ni \alpha \mapsto V^\alpha \in \mathcal{C}_{\hat{\mathrm{g}}}$.

Take $\{V^{\alpha_i}\}_{i \in J}$ as the given data $\rightsquigarrow \mathcal{F}\colon \bigoplus_\beta R(\beta)\text{-mod}^0 \to \mathcal{C}_{\hat{\mathrm{g}}}$.

∘ In this case, $R(\beta)$ is of type g $\Rightarrow \mathcal{F}$ is exact.

∘ The image of $\mathcal{F}\colon \bigoplus_\beta R(\beta)\text{-mod}^0 \to \mathcal{C}_{\hat{\mathrm{g}}}$ is contained in $\mathcal{C}_Q$.

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Theorem ([KKK15])

In this case, the gene'd QASW duality functor $\mathcal{F}\colon \bigoplus_\beta R(\beta)\text{-mod}^0 \to \mathcal{C}_Q$, which is monoidal and exact, gives one-to-one corresp. between simples. ($\Rightarrow \bigoplus_\beta K(R(\beta)\text{-mod}^0) \xrightarrow{\sim} K(\mathcal{C}_Q)$)

$\mathcal{F}\colon \bigoplus_\beta R(\beta)\text{-mod}^0 \to \mathcal{C}_Q$ is monoidal, exact, gives one-to-one corresp. between simples.

Natrual problems

(i) Is this an equivalence?

(ii) Is there a generalization to the cases other than untwisted ADE types?

Both problems have been solved affirmatively!

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Theorem ([Fujita, 17], [Fujita, 20])

The gene'd QASW duality functor $\mathcal{F}\colon \bigoplus_\beta R(\beta)\text{-mod}^0 \to \mathcal{C}_Q$ gives an equivalence of monoidal categories (in untwisted ADE types).

In the proof of [Fujita, 17], he used the geometric representation theory on quiver varieties and the theory of affine highest weight categories (we will return to this result later).

generalization to non-ADE cases

$\mathfrak{g}$: non-simply laced (untwisted or twisted) affine Lie algebra

Set a simple Lie algebra g to be as follows:

$U_q'(\mathfrak{g})$	$B_n^{(1)}$	$C_n^{(1)}$	$F_4^{(1)}$	$G_2^{(1)}$	$A_n^{(2)}$	$D_n^{(2)}$	$E_6^{(2)}$	$D_4^{(3)}$
g	A_{2n-1}	D_{n+1}	E_6	D_4	A_n	D_n	E_6	D_4

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g	A_{2n-1}	D_{n+1}	E_6	D_4	A_n	D_n	E_6	D_4

Similarly as ADE cases, define a map $R^+_{\mathrm{g}} \ni \alpha \mapsto V^\alpha \in \mathcal{C}_{\mathfrak{g}}$,

and set $\mathcal{C}_Q = \langle V^\alpha \rangle_{\otimes,\text{ext.,subquot.}}$

$\rightsquigarrow$ functor $\mathcal{F}$: $\bigoplus_\beta R(\beta)\text{-mod}^0 \to \mathcal{C}_Q$ $\quad$ ($\{R(\beta)\}$: quiver Hecke of type g)

Theorem ([KKK16], [Kashiwara–Oh, 19], [Oh–Scrimshaw, 19])

In all the above cases, the gene'd QASW duality functor $\mathcal{F}$ is monoidal, exact, and gives one-to-one correspondence between simple modules.

$\left(\Rightarrow \bigoplus_\beta K(R(\beta)\text{-mod}^0) \xrightarrow{\sim} K(\mathcal{C}_Q).\right)$

Summary

$U_q'(\mathfrak{g})$	monoidal	exact	bij. of simples	equiv.
ADE	◯	◯	◯	◯
others	◯	◯	◯	?

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$U_q'(\mathfrak{g})$	monoidal	exact	bij. of simples	equiv.
ADE	◯	◯	◯	◯
others	◯	◯	◯	?

Theorem ([N])

In general types, the gene'd QASW duality functor $\mathcal{F}$ gives an equivalence of monoidal categories $\bigoplus_\beta R(\beta)\text{-mod}^0$ and $\mathcal{C}_Q$.

Proof to [Conjecture 5.7, KKK16], [Conjecture 6.11, KO19].

Corollary

Let $\mathfrak{g}^{(1)}$: untwisted ADE, $\mathfrak{g}^{(t)}$: twisted, ${}^{L}\mathfrak{g}^{(t)}$: the Langland dual of $\mathfrak{g}^{(t)}$

$\mathcal{C}_{Q^{(1)}}$, $\mathcal{C}_{Q^{(t)}}$, $\mathcal{C}_{{}^LQ}$: corresponding (generalizations) of HL subcategories

Corollary

The monoidal categories $\mathcal{C}_{Q^{(1)}}$, $\mathcal{C}_{Q^{(t)}}$, $\mathcal{C}_{{}^LQ}$ are mutually equivalent.

$\because$ The corresponding quiver Hecke algebras $R(\beta)$ are the same. □

Ex.

$$\begin{array}{ccccc} & & \mathcal{C}_{Q^{(1)}} \subseteq \mathcal{C}_{A^{(1)}_{2n-1}} & & \\ & & \uparrow \wr & & \\ \mathcal{C}_{A^{(2)}_{2n-1}} \supseteq \mathcal{C}_{Q^{(2)}} & \xleftarrow{\sim} & \bigoplus_{\beta} R^{A_{2n-1}}(\beta)\text{-mod}^0 & \xrightarrow{\sim} & \mathcal{C}_{{}^LQ} \subseteq \mathcal{C}_{B^{(1)}_n} \end{array}$$

strategy of the proof of $\mathcal{F}\colon \bigoplus R(\beta)\text{-mod}^0 \xrightarrow{\sim} \mathcal{C}_Q$

<u>Fact</u> $\mathcal{C}_Q$ has a block dec. $\mathcal{C}_Q = \bigoplus_\beta \mathcal{C}_{Q,\beta}$ $(\beta \in Q^+)$ such that

$\mathcal{F}_\beta\colon R(\beta)\text{-mod}^0 \to \mathcal{C}_{Q,\beta}$

$\therefore$ Enough to prove $\mathcal{F}_\beta\colon R(\beta)\text{-mod}^0 \xrightarrow{\sim} \mathcal{C}_{Q,\beta}$ for each β.

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From the homological viewpoint, $R(\beta)\text{-mod}^0$ and $\mathcal{C}_{Q,\beta}$ are too small (e.g., not enough proj.)

$R(\beta) = \bigoplus_{n\in\mathbb{Z}} R(\beta)_n \rightsquigarrow \widehat{R}(\beta) = \prod_n R(\beta)_n$: completion (cf. $\mathbb{C}[z] \rightsquigarrow \mathbb{C}[\![z]\!]$), and consider $\widehat{R}(\beta)\text{-Mod}$ instead.

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advantage $\circ$ $\widehat{R}(\beta)\text{-mod} = R(\beta)\text{-mod}^0$

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advantage
- $\widehat{R}(\beta)\text{-mod} = R(\beta)\text{-mod}^0$
- $\widehat{R}(\beta)\text{-Mod}$ is **affine highest weight category**!

(a generalization of highest weight cat. by Cline–Parshall–Scott.

$\Delta(\lambda)$: standard $\twoheadrightarrow L(\lambda)$: simple $\hookrightarrow \overline{\nabla}(\lambda)$: proper costandard)

$$\mathcal{F}_\beta \colon R(\beta)\text{-mod}^0 \to \mathcal{C}_{Q,\beta}, \qquad M \mapsto \widehat{V}^{\otimes\beta} \otimes_{R(\beta)} M$$

$(\widehat{V}^{\otimes\beta}\colon (U_q'(\mathfrak{g}), R(\beta))$-bimod.)

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$$\begin{array}{c}\cap \\ \widehat{R}(\beta)\text{-Mod} \\ \text{(aff. h.w.)}\end{array}$$

$(\widehat{V}^{\otimes\beta} \colon (U_q'(\mathfrak{g}), R(\beta))\text{-bimod.})$

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extend $\downarrow$ $\quad\cap$

$$\mathcal{F}_\beta \colon \underset{\text{(aff. h.w.)}}{\widehat{R}(\beta)\text{-Mod}} \to \{U_q'(\mathfrak{g})\text{-modules}\} \quad (\widehat{V}^{\otimes\beta} \colon (U_q'(\mathfrak{g}), \widehat{R}(\beta))\text{-bimod.})$$

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(aff. h.w.)

Theorem ([Fujita, 18])

A_i-Mod: affine h.w. $(i = 1, 2)$, $F \colon A_1\text{-Mod} \to A_2\text{-Mod}$: exact.

Assume (i) A_i is finitely generated over its center $(i = 1, 2)$,

(ii) $^\exists$bijection $f \colon \Pi_1 \to \Pi_2$ such that $F(\Delta(\pi)) = \Delta(f(\pi))$,

$F(\overline{\nabla}(\pi)) = \overline{\nabla}(f(\pi))$ for $^\forall\pi$.

Then F is an equivalence.

We consider the following project:

(i) Find an algebra A with an algebra homomorphism $\Phi\colon U_q'(\mathfrak{g}) \to A$.

(ii) Show that $\Phi^*|_{A\text{-mod}}\colon A\text{-mod} \to U_q'(\mathfrak{g})\text{-mod}$ gives an equivalence between A-mod and $\mathcal{C}_{Q,\beta}$.

(iii) Define $\mathcal{F}'_\beta\colon \widehat{R}(\beta)\text{-Mod} \to A\text{-Mod}$ s.t. $\Phi^* \circ \mathcal{F}'_\beta|_{\widehat{R}(\beta)\text{-mod}} = \mathcal{F}_\beta$.

(iv) Show that A-Mod is aff. h.w., and $\mathcal{F}'_\beta$ gives an equivalence $\mathcal{F}'_\beta\colon \widehat{R}(\beta)\text{-Mod} \xrightarrow{\sim} A\text{-Mod}$.

$$\begin{array}{ccccc} \mathcal{F}_\beta\colon \widehat{R}(\beta)\text{-mod} & \to & A\text{-mod} & \overset{\Phi^*}{\xrightarrow{\sim}} & \mathcal{C}_{Q,\beta} \\ \cap & & \cap & & \\ \mathcal{F}'_\beta\colon \widehat{R}(\beta)\text{-Mod} & \to & A\text{-Mod} & & \end{array}$$

(aff. h.w.)

proof in untwisted ADE in [Fujita, 17]

(i) Find an algebra A with an algebra homomorphism $\Phi\colon U_q'(\mathfrak{g}) \to A$.

(ii) Show that $\Phi^*|_{A\text{-mod}}\colon A\text{-mod} \to U_q'(\mathfrak{g})\text{-mod}$ gives an equivalence between A-mod and $\mathcal{C}_Q$.

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In [Fujita, 17], he achieved this project with $A = \widehat{\mathcal{K}}^{\mathbb{G}}(Z^\bullet)$ (completed equiv. K-gps of the Steinberg type graded quiver var.)

(i) ${}^\exists\Phi\colon U_q'(\mathfrak{g}) \to \widehat{\mathcal{K}}^{\mathbb{G}}(Z^\bullet)$ by Nakajima,

(iii) define $\widehat{\mathcal{K}}^{\mathbb{G}}(Z^\bullet) \curvearrowright \widehat{V}^{\otimes\beta}$ geometrically,

(ii), (iv) work hard (omit)

proof in general types

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<u>recall</u> $\widehat{V}^{\otimes\beta}$: $(U_q'(\mathfrak{g}), \widehat{R}(\beta))$-bimod., $\mathcal{F}_\beta(M) := \widehat{V}^{\otimes\beta} \otimes_{\widehat{R}(\beta)} M$

Set $\mathbb{E}^\beta = \mathrm{End}_{\widehat{R}(\beta)^{\mathrm{opp}}}(\widehat{V}^{\otimes\beta})$ (**analog of Schur algebra**).

This $\mathbb{E}^\beta$ is our A. (i), (iii) are obvious.

Theorem ([N])

Set $\mathbb{E}^\beta = \mathrm{End}_{\widehat{R}(\beta)^{\mathrm{opp}}}(\widehat{V}^{\otimes\beta})$.

(i) The alg. hom. $\Phi\colon U_q'(\mathfrak{g}) \to \mathbb{E}^\beta$ induces an equiv. $\Phi^*\colon \mathbb{E}^\beta\text{-mod} \xrightarrow{\sim} \mathcal{C}_{Q,\beta}$.

(ii) $\mathbb{E}^\beta\text{-Mod}$ is aff. h.w., and $\mathcal{F}_\beta$ gives an equiv. $\widehat{R}(\beta)\text{-Mod} \xrightarrow{\sim} \mathbb{E}^\beta\text{-Mod}$.

In the proof, we use **affine cellular str.** of (a quotient) of $U_q'(\mathfrak{g})$ and $\mathbb{E}^\beta$.

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future work Study (a polynomial part of) $\mathbb{E}^\beta$.

- ◦ generators and relations?
- ◦ Is this graded?
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Thank you for your attention!