Generalization of the extended T-systems via strong duality data

Katsuyuki Naoi

Tokyo University of Agriculture and Technology

16th International Symposium on Natural Sciences,

October 10, 2024

Plan

Main Theorem

Mukhin–Young's extended T-systems are generalized to affine cuspidal modules defined from a strong duality datum of type A.

1. Mukhin–Young's extended T-systems (what we generalize)
2. Strong duality data and affine cuspidal modules (how we generalize)
3. Main Theorem
4. Proof (relations between the extended T-systems and Kashiwara crystals)

Plan

Main Theorem

Mukhin–Young's extended T-systems are generalized to affine cuspidal modules defined from a strong duality datum of type A.

1. Mukhin–Young's extended T-systems (what we generalize)
2. Strong duality data and affine cuspidal modules (how we generalize)
3. Main Theorem
4. Proof (relations between the extended T-systems and Kashiwara crystals)

Notations

$\mathfrak{g}$: complex affine Lie algebra (e.g. $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}[t^{\pm 1}] \oplus \mathbb{C}K$)

$U_q'(\mathfrak{g})$: **quantum affine algebra** with index set $[0, n]$ and $q \in \mathbb{C}^\times$ not root of 1

(associative algebra over $\mathbb{C}$ such that "$\lim_{q \to 1} U_q'(\mathfrak{g}) = U(\mathfrak{g})$")

$\mathscr{C}_\mathfrak{g}$: the cat. of **finite-dimensional** $U_q'(\mathfrak{g})$-mod. (of type 1)

- $\mathscr{C}_\mathfrak{g}$ is a monoidal category with $\otimes$ and the trivial module $\mathbf{1}$
  $\Rightarrow K(\mathscr{C}_\mathfrak{g})$ has a ring structure (**Grothendieck ring**)
- Each $M \in \mathscr{C}_\mathfrak{g}$ has the right dual $\mathscr{D}(M)$ and the left dual $\mathscr{D}^{-1}(M)$

fundamental problem For simple $M, N \in \mathscr{C}_\mathfrak{g}$, determine the structure of $M \otimes N$.

- $\mathscr{C}_\mathfrak{g}$ is quite complicated (e.g., not semisimple, $M \otimes N \not\simeq N \otimes M$ in general), and it is extremely difficult to solve this problem for general M, N.

Notations

$\mathfrak{g}$: complex affine Lie algebra (e.g. $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}[t^{\pm 1}] \oplus \mathbb{C}K$)

$U_q'(\mathfrak{g})$: **quantum affine algebra** with index set $[0, n]$ and $q \in \mathbb{C}^\times$ not root of 1

(associative algebra over $\mathbb{C}$ such that "$\lim_{q \to 1} U_q'(\mathfrak{g}) = U(\mathfrak{g})$")

$\mathscr{C}_\mathfrak{g}$: the cat. of **finite-dimensional** $U_q'(\mathfrak{g})$-mod. (of type 1)

- $\mathscr{C}_\mathfrak{g}$ is a monoidal category with $\otimes$ and the trivial module $\mathbf{1}$
  $\Rightarrow K(\mathscr{C}_\mathfrak{g})$ has a ring structure (**Grothendieck ring**)
- Each $M \in \mathscr{C}_\mathfrak{g}$ has the right dual $\mathscr{D}(M)$ and the left dual $\mathscr{D}^{-1}(M)$

fundamental problem For simple $M, N \in \mathscr{C}_\mathfrak{g}$, determine the structure of $M \otimes N$.

- $\mathscr{C}_\mathfrak{g}$ is quite complicated (e.g., not semisimple, $M \otimes N \not\simeq N \otimes M$ in general), and it is extremely difficult to solve this problem for general M, N.

Notations

$\mathfrak{g}$: complex affine Lie algebra (e.g. $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}[t^{\pm 1}] \oplus \mathbb{C}K$)

$U_q'(\mathfrak{g})$: **quantum affine algebra** with index set $[0, n]$ and $q \in \mathbb{C}^\times$ not root of 1

(associative algebra over $\mathbb{C}$ such that "$\lim_{q \to 1} U_q'(\mathfrak{g}) = U(\mathfrak{g})$")

$\mathscr{C}_\mathfrak{g}$: the cat. of **finite-dimensional** $U_q'(\mathfrak{g})$-mod. (of type 1)

- $\mathscr{C}_\mathfrak{g}$ is a monoidal category with $\otimes$ and the trivial module $\mathbf{1}$
  $\Rightarrow K(\mathscr{C}_\mathfrak{g})$ has a ring structure (**Grothendieck ring**)
- Each $M \in \mathscr{C}_\mathfrak{g}$ has the right dual $\mathscr{D}(M)$ and the left dual $\mathscr{D}^{-1}(M)$

fundamental problem For simple $M, N \in \mathscr{C}_\mathfrak{g}$, determine the structure of $M \otimes N$.

- $\mathscr{C}_\mathfrak{g}$ is quite complicated (e.g., not semisimple, $M \otimes N \not\simeq N \otimes M$ in general), and it is extremely difficult to solve this problem for general M, N.

Notations

$\mathfrak{g}$: complex affine Lie algebra (e.g. $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}[t^{\pm 1}] \oplus \mathbb{C}K$)

$U_q'(\mathfrak{g})$: **quantum affine algebra** with index set $[0, n]$ and $q \in \mathbb{C}^\times$ not root of 1

(associative algebra over $\mathbb{C}$ such that "$\lim_{q \to 1} U_q'(\mathfrak{g}) = U(\mathfrak{g})$")

$\mathscr{C}_\mathfrak{g}$: the cat. of **finite-dimensional** $U_q'(\mathfrak{g})$-mod. (of type 1)

- $\mathscr{C}_\mathfrak{g}$ is a monoidal category with $\otimes$ and the trivial module $\mathbf{1}$
  $\Rightarrow K(\mathscr{C}_\mathfrak{g})$ has a ring structure (**Grothendieck ring**)
- Each $M \in \mathscr{C}_\mathfrak{g}$ has the right dual $\mathscr{D}(M)$ and the left dual $\mathscr{D}^{-1}(M)$

<u>fundamental problem</u> For simple $M, N \in \mathscr{C}_\mathfrak{g}$, determine the structure of $M \otimes N$.

- $\mathscr{C}_\mathfrak{g}$ is quite complicated (e.g., not semisimple, $M \otimes N \not\cong N \otimes M$ in general), and it is extremely difficult to solve this problem for general M, N.

Theorem (Chari-Pressley, 95)

$$\{\text{simples in } \mathscr{C}_{\mathfrak{g}}\}/_{\cong} \overset{1:1}{\leftrightarrow} \{\boldsymbol{\pi}(u) = (\pi_1(u), \dots, \pi_n(u)) \mid \pi_i(u) \in 1 + u\mathbb{C}[u]\}.$$

Drinfeld polynomials

For $i \in [1, n]$ and $k \in \mathbb{Z}$, set

$Y_{i,k} = Y_{i,k}(u) := (1, \dots, 1 - q^k u, \dots, 1) \in (1 + u\mathbb{C}[u])^{\times n}$

$\rightsquigarrow$ For a sequence $((i_1, k_1), \dots, (i_p, k_p)) \in ([1, n] \times \mathbb{Z})^{\times p}$,

$\prod_{r=1}^{p} Y_{i_r,k_r} = (\pi_1(u), \dots, \pi_p(u))$, where $\pi_i(u) = \prod_{r; i_r = i} (1 - q^{k_r} u)$

$\rightsquigarrow$ simple module $L(\prod_{r=1}^{p} Y_{i_1,k_1})$ is defined (**monomial parametrization**)

A simple module $L(Y_{i,k})$ is called a **fundamental module**.

Theorem (Chari-Pressley, 95)

$$\{\text{simples in } \mathscr{C}_{\mathfrak{g}}\}/_{\cong} \overset{1:1}{\leftrightarrow} \{\boldsymbol{\pi}(u) = (\pi_1(u), \ldots, \pi_n(u)) \mid \pi_i(u) \in 1 + u\mathbb{C}[u]\}.$$

Drinfeld polynomials

For $i \in [1, n]$ and $k \in \mathbb{Z}$, set

$$Y_{i,k} = Y_{i,k}(u) := (1, \ldots, 1 - q^k u, \ldots, 1) \in (1 + u\mathbb{C}[u])^{\times n}$$

$\rightsquigarrow$ For a sequence $\big((i_1, k_1), \ldots, (i_p, k_p)\big) \in ([1, n] \times \mathbb{Z})^{\times p}$,

$$\prod_{r=1}^{p} Y_{i_r, k_r} = \big(\pi_1(u), \ldots, \pi_p(u)\big), \text{ where } \pi_i(u) = \prod_{r; i_r = i} (1 - q^{k_r} u)$$

$\rightsquigarrow$ simple module $L(\prod_{r=1}^{p} Y_{i_1, k_1})$ is defined (**monomial parametrization**)

A simple module $L(Y_{i,k})$ is called a **fundamental module**.

T-systems

For general affine $\mathfrak{g}$, the **T-systems** are certain relations in $K(\mathscr{C}_{\mathfrak{g}})$ for the tensor product of Kirillov-Reshetikhin (KR) modules $L(\prod_{k=1}^{p} Y_{i,r+2d_ik})$:

Ex. (T-systems for untwisted, simply-laced $\mathfrak{g}$)

$$\Big[L\Big(\prod_{k=1}^{p-1} Y_{i,r+2k}\Big) \otimes L\Big(\prod_{k=2}^{p} Y_{i,r+2k}\Big)\Big]$$
$$= \Big[L\Big(\prod_{k=1}^{p} Y_{i,r+2k}\Big) \otimes L\Big(\prod_{k=2}^{p-1} Y_{i,r+2k}\Big)\Big] + \Big[\bigotimes_{c_{ij}=-1} L\Big(\prod_{k=1}^{p-1} Y_{j,r+2k+1}\Big)\Big]$$

For $\mathfrak{g}$ of type $A_n^{(1)}$ and $B_n^{(1)}$, Mukhin and Young introduced in '12 similar relations (**extended T-systems**) for **prime snake modules** (which we will recall next). These contain all T-systems of these types.

T-systems

For general affine $\mathfrak{g}$, the **T-systems** are certain relations in $K(\mathscr{C}_{\mathfrak{g}})$ for the tensor product of Kirillov-Reshetikhin (KR) modules $L(\prod_{k=1}^{p} Y_{i,r+2d_ik})$:

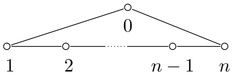
Ex. (T-systems for untwisted, simply-laced $\mathfrak{g}$)

$$\Big[L\Big(\prod_{k=1}^{p-1} Y_{i,r+2k}\Big) \otimes L\Big(\prod_{k=2}^{p} Y_{i,r+2k}\Big)\Big]$$
$$= \Big[L\Big(\prod_{k=1}^{p} Y_{i,r+2k}\Big) \otimes L\Big(\prod_{k=2}^{p-1} Y_{i,r+2k}\Big)\Big] + \Big[\bigotimes_{c_{ij}=-1} L\Big(\prod_{k=1}^{p-1} Y_{j,r+2k+1}\Big)\Big]$$

For $\mathfrak{g}$ of type $A_n^{(1)}$ and $B_n^{(1)}$, Mukhin and Young introduced in '12 similar relations (**extended T-systems**) for **prime snake modules** (which we will recall next). These contain all T-systems of these types.

Snake modules in type $A_n^{(1)}$

Assume $\mathfrak{g}$ is of type $A_n^{(1)}$:



Set $J_n := \{(i,k) \mid k \equiv i \pmod 2\} \subseteq [1,n] \times \mathbb{Z}$

$(n = 5)$

$(i \setminus k)$	$\cdots$	0	1	2	3	4	5	6	7	$\cdots$
1			○		○		○		○	
2		○		○		○		○		
3			○		○		○		○	
4		○		○		○		○		
5			○		○		○		○	

Definition

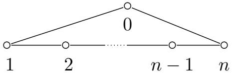
A sequence $\xi = ((i_1,k_1),\dots,(i_p,k_p)) \in J_n^p$ is a **snake**

$\overset{\text{def}}{\Leftrightarrow}$ for $1 \le {}^{\forall} r < p$, setting $(i,k) = (i_r,k_r)$ and $(i',k') = (i_{r+1},k_{r+1})$

$$|i - i'| + 2 \le k - k'$$

Snake modules in type $A_n^{(1)}$

Assume $\mathfrak{g}$ is of type $A_n^{(1)}$:



Set $J_n := \{(i,k) \mid k \equiv i \pmod 2\} \subseteq [1,n] \times \mathbb{Z}$

$(n = 5)$

$(i \setminus k)$	$\cdots$	0	1	2	3	4	5	6	7	$\cdots$
1			$\circ$		$\circ$		$\circ$		$\circ$	
2		$\circ$		$\circ$		$\circ$		$\circ$		
3			$\circ$		$\circ$		$\circ$		$\circ$	
4		$\circ$		$\circ$		$\circ$		$\circ$		
5			$\circ$		$\circ$		$\circ$		$\circ$	

Definition

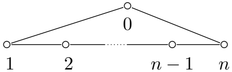
A sequence $\xi = ((i_1,k_1),\ldots,(i_p,k_p)) \in J_n^p$ is a **snake** (prime snake)

$\overset{\text{def}}{\Leftrightarrow}$ for $1 \le {}^\forall r < p$, setting $(i,k) = (i_r,k_r)$ and $(i',k') = (i_{r+1},k_{r+1})$,

$$|i - i'| + 2 \le k - k' \ (\le \min\{i + i', 2n + 2 - i - i'\})$$

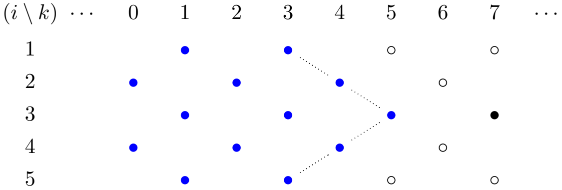
Snake modules in type $A_n^{(1)}$

Assume $\mathfrak{g}$ is of type $A_n^{(1)}$:



Set $J_n := \{(i,k) \mid k \equiv i \pmod 2\} \subseteq [1,n] \times \mathbb{Z}$

$(n = 5)$



Definition

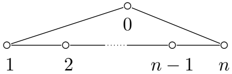
A sequence $\xi = ((i_1,k_1),\ldots,(i_p,k_p)) \in J_n^p$ is a **snake** (prime snake)

$\overset{\text{def}}{\Leftrightarrow}$ for $1 \le {}^\forall r < p$, setting $(i,k) = (i_r,k_r)$ • and $(i',k') = (i_{r+1},k_{r+1})$ •,

$$|i - i'| + 2 \le k - k' \; (\le \min\{i + i', 2n + 2 - i - i'\})$$

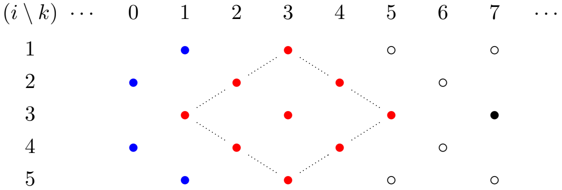
Snake modules in type $A_n^{(1)}$

Assume $\mathfrak{g}$ is of type $A_n^{(1)}$:



Set $J_n := \{(i,k) \mid k \equiv i \pmod 2\} \subseteq [1,n] \times \mathbb{Z}$

$(n = 5)$



Definition

A sequence $\boldsymbol{\xi} = \big((i_1,k_1),\dots,(i_p,k_p)\big) \in J_n^p$ is a **snake** (**prime snake**)

$\overset{\text{def}}{\Leftrightarrow}$ for $1 \le {}^\forall r < p$, setting $(i,k) = (i_r,k_r)$ • and $(i',k') = (i_{r+1},k_{r+1})$ •,

$$|i - i'| + 2 \le k - k' \;(\le \min\{i + i', 2n + 2 - i - i'\})$$

$(i \setminus k)$	$\cdots$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	$\cdots$
1			○		○		○		○		○		○		○	
2		●		●		○		○		○		○		○		
3			○		○		●		○		○		○		○	
4		○		○		○		○		●		○		○		
5			○		○		○		○		○		●		●	

$\boldsymbol{\xi} = ((i_1, k_1), \ldots, (i_p, k_p))$: snake $\Rightarrow$ $L(\boldsymbol{\xi}) = L(\prod_{r=1}^{p} Y_{i_r,k_r})$: **snake module**

prime snake $\xi \rightsquigarrow$ two **neighboring snakes** ξ_{H} $(\star)$, ξ_{L} $(\star)$

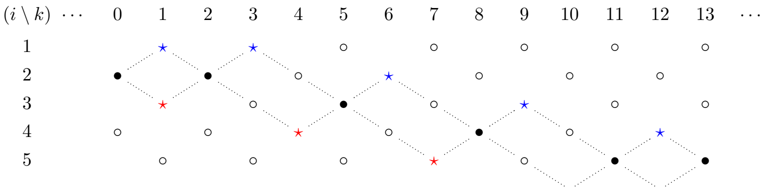
Theorem (MY12)

- ξ: prime $\Leftrightarrow$ $L(\xi)$: prime (i.e. $L(\xi) \cong M \otimes N \Rightarrow M \cong \mathbf{1}$ or $N \cong \mathbf{1}$)
- $[L(\prod_{r=1}^{p-1} Y_{i_r,k_r}) \otimes L(\prod_{r=2}^{p} Y_{i_r,k_r})] = [L(\xi) \otimes L(\prod_{r=2}^{p-1} Y_{i_r,k_r})] + [L(\xi_{\mathrm{H}}) \otimes L(\xi_{\mathrm{L}})]$

  (the last two factors: simple)

Rem. KR module $\Leftrightarrow$ straight snake ● ● ● $\cdots$

- $B_n^{(1)}$ case is similar but slightly more complicated (omit).



$\xi = ((i_1, k_1), \ldots, (i_p, k_p))$: snake $\Rightarrow$ $L(\xi) = L(\prod_{r=1}^{p} Y_{i_r,k_r})$: **snake module**

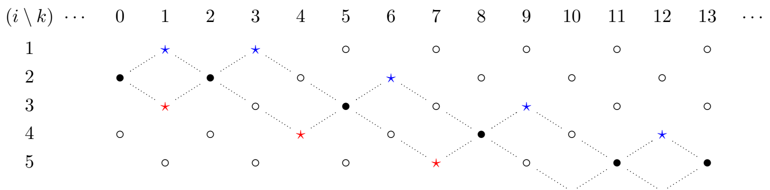
prime snake $\xi \rightsquigarrow$ two **neighboring snakes** ξ_{H} ($\star$), ξ_{L} ($\star$)

Theorem (MY12)

- ξ: prime $\Leftrightarrow$ $L(\xi)$: prime (i.e. $L(\xi) \cong M \otimes N \Rightarrow M \cong \mathbf{1}$ or $N \cong \mathbf{1}$)
- $[L(\prod_{r=1}^{p-1} Y_{i_r,k_r}) \otimes L(\prod_{r=2}^{p} Y_{i_r,k_r})] = [L(\xi) \otimes L(\prod_{r=2}^{p-1} Y_{i_r,k_r})] + [L(\xi_{\mathrm{H}}) \otimes L(\xi_{\mathrm{L}})]$ (simple)

Rem. KR module $\Leftrightarrow$ straight snake • • • ⋯

- $B_n^{(1)}$ case is similar but slightly more complicated (omit).



$\boldsymbol{\xi} = ((i_1, k_1), \dots, (i_p, k_p))$: snake $\Rightarrow$ $L(\boldsymbol{\xi}) = L(\prod_{r=1}^{p} Y_{i_r,k_r})$: **snake module**

prime snake $\boldsymbol{\xi} \rightsquigarrow$ two **neighboring snakes** $\boldsymbol{\xi}_{\mathrm{H}}$ ($\star$), $\boldsymbol{\xi}_{\mathrm{L}}$ ($\star$)

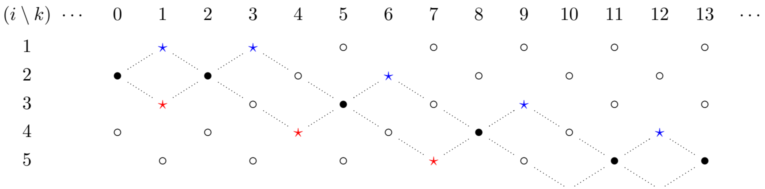
Theorem (MY12)

- $\boldsymbol{\xi}$: prime $\Leftrightarrow$ $L(\boldsymbol{\xi})$: prime (i.e. $L(\boldsymbol{\xi}) \cong M \otimes N \Rightarrow M \cong \mathbf{1}$ or $N \cong \mathbf{1}$)
- $[L(\prod_{r=1}^{p-1} Y_{i_r,k_r}) \otimes L(\prod_{r=2}^{p} Y_{i_r,k_r})] = [L(\boldsymbol{\xi}) \otimes L(\prod_{r=2}^{p-1} Y_{i_r,k_r})] + [L(\boldsymbol{\xi}_{\mathrm{H}}) \otimes L(\boldsymbol{\xi}_{\mathrm{L}})]$

  simple

Rem. KR module $\Leftrightarrow$ straight snake • • • ···

- $B_n^{(1)}$ case is similar but slightly more complicated (omit).



$\boldsymbol{\xi} = ((i_1, k_1), \ldots, (i_p, k_p))$: snake $\Rightarrow$ $L(\boldsymbol{\xi}) = L(\prod_{r=1}^{p} Y_{i_r,k_r})$: **snake module**

prime snake $\boldsymbol{\xi} \rightsquigarrow$ two **neighboring snakes** $\boldsymbol{\xi}_{\mathrm{H}}$ ($\star$), $\boldsymbol{\xi}_{\mathrm{L}}$ ($\star$)

Theorem (MY12)

- $\boldsymbol{\xi}$: prime $\Leftrightarrow$ $L(\boldsymbol{\xi})$: prime (i.e. $L(\boldsymbol{\xi}) \cong M \otimes N \Rightarrow M \cong \mathbf{1}$ or $N \cong \mathbf{1}$)
- $[L(\prod_{r=1}^{p-1} Y_{i_r,k_r}) \otimes L(\prod_{r=2}^{p} Y_{i_r,k_r})] = [L(\boldsymbol{\xi}) \otimes L(\prod_{r=2}^{p-1} Y_{i_r,k_r})] + [L(\boldsymbol{\xi}_{\mathrm{H}}) \otimes L(\boldsymbol{\xi}_{\mathrm{L}})]$

  (the last two factors: simple)

Rem. KR module $\Leftrightarrow$ straight snake • • • ···

- $B_n^{(1)}$ case is similar but slightly more complicated (omit).

Extended T-systems

$$[L(\prod_{r=1}^{p-1} Y_{i_r,k_r}) \otimes L(\prod_{r=2}^{p} Y_{i_r,k_r})] = [L(\boldsymbol{\xi}) \otimes L(\prod_{r=2}^{p-1} Y_{i_r,k_r})] + [L(\boldsymbol{\xi}_{\mathrm{H}}) \otimes L(\boldsymbol{\xi}_{\mathrm{L}})]$$

This was proved by showing that the **q-characters** of both sides coincide.

(inj. algebra hom. $\chi_q \colon K(\mathscr{C}_{\mathfrak{g}}) \hookrightarrow \mathbb{Z}[Y_{i,a}^{\pm 1} \mid i \in [1,n], a \in \mathbb{C}^{\times}]$)

Goal

We will show that the same relations also hold if we replace the snake modules with simple modules of the form $\mathrm{hd}(S_{j_1} \otimes \cdots \otimes S_{j_p})$.

($\mathrm{hd}\, M :=$ the maximal semisimple quotient of M)

Here the modules S_j are **affine cuspidal modules**, which are defined from a **strong duality datum** (recalled next). These notions are introduced by Kashiwara–Kim–Oh–Park.

Plan

1. Mukhin–Young's extended T-systems (what we generalize)
2. Strong duality data and affine cuspidal modules introduced by Kashiwara–Kim–Oh–Park
3. Main Theorem
4. Proof (relations between the extended T-systems and Kashiwara crystals)

invariants $\mathfrak{d}(M, N)$

For simple mod. $M, N \in \mathscr{C}_\mathfrak{g}$, an isom. $M \otimes N \xrightarrow{\sim} N \otimes M$ doesn't necessarily exist, but it is known that there is an isom.

$$R^{\mathrm{norm}}_{M,N} \colon \mathbb{C}(z) \otimes_{\mathbb{C}[z^{\pm 1}]} (M \otimes N[z^{\pm 1}]) \xrightarrow{\sim} \mathbb{C}(z) \otimes_{\mathbb{C}[z^{\pm 1}]} (N[z^{\pm 1}] \otimes M)$$

(normalized R-matrix)

Definition

For simple modules M, N of $\mathscr{C}_\mathfrak{g}$, define $\mathfrak{d}(M, N) \in \mathbb{Z}_{\geq 0}$ by

$\mathfrak{d}(M, N) :=$

(deg. of the pole of $R^{\mathrm{norm}}_{M,N}$ at $z = 1$) + (deg. of the pole of $R^{\mathrm{norm}}_{N,M}$ at $z = 1$)

It is known that $\mathfrak{d}(M, N) = 0 \Leftrightarrow M \otimes N \cong N \otimes M$ (under mild conditions).

Intuitively, $\mathfrak{d}(M, N)$ measures how far from the existence of $M \otimes N \xrightarrow{\sim} N \otimes M$.

invariants $\mathfrak{d}(M, N)$

For simple mod. $M, N \in \mathscr{C}_{\mathfrak{g}}$, an isom. $M \otimes N \xrightarrow{\sim} N \otimes M$ doesn't necessarily exist, but it is known that there is an isom.

$R^{\mathrm{norm}}_{M,N} \colon \mathbb{C}(z) \otimes_{\mathbb{C}[z^{\pm 1}]} (M \otimes N[z^{\pm 1}]) \xrightarrow{\sim} \mathbb{C}(z) \otimes_{\mathbb{C}[z^{\pm 1}]} (N[z^{\pm 1}] \otimes M)$

(normalized R-matrix)

Definition

For simple modules M, N of $\mathscr{C}_{\mathfrak{g}}$, define $\mathfrak{d}(M, N) \in \mathbb{Z}_{\geq 0}$ by

$\mathfrak{d}(M, N) :=$
(deg. of the pole of $R^{\mathrm{norm}}_{M,N}$ at $z = 1$) $+$ (deg. of the pole of $R^{\mathrm{norm}}_{N,M}$ at $z = 1$)

It is known that $\mathfrak{d}(M, N) = 0 \Leftrightarrow M \otimes N \cong N \otimes M$ (under mild conditions).

Intuitively, $\mathfrak{d}(M, N)$ measures how far from the existence of $M \otimes N \xrightarrow{\sim} N \otimes M$.

strong duality data

Fix a Cartan matrix $C = (c_{ij})_{i,j\in I}$ of finite ADE type (irrelevant to the type of $\mathfrak{g}$)

Definition

A family of real simple modules $\mathcal{D} = \{L_i\}_{i\in I} \subseteq \mathscr{C}_{\mathfrak{g}}$ is called a **strong duality datum** (associated with C) if

1. $\mathfrak{d}(L_i, \mathscr{D}^k L_i) = \delta_{k,0} \quad ({}^\forall i \in I, {}^\forall k \in \mathbb{Z})$,
2. $\mathfrak{d}(L_i, \mathscr{D}^k L_j) = -c_{ij}(\delta_{k,1} + \delta_{k,-1}) \quad (i \neq j, {}^\forall k \in \mathbb{Z})$.

Recall that simple M is real $\overset{\text{def}}{\Leftrightarrow}$ $M \otimes M$: simple.

strong duality data

Fix a Cartan matrix $C = (c_{ij})_{i,j \in I}$ of finite ADE type (irrelevant to the type of $\mathfrak{g}$)

Definition

A family of real simple modules $\mathcal{D} = \{L_i\}_{i \in I} \subseteq \mathscr{C}_{\mathfrak{g}}$ is called a **strong duality datum** (associated with C) if

1. $\mathfrak{d}(L_i, \mathscr{D}^k L_i) = \delta_{k,0} \quad ({}^{\forall} i \in I, {}^{\forall} k \in \mathbb{Z})$,
2. $\mathfrak{d}(L_i, \mathscr{D}^k L_j) = -c_{ij}(\delta_{k,1} + \delta_{k,-1}) \quad (i \neq j, {}^{\forall} k \in \mathbb{Z})$.

Recall that simple M is real $\overset{\text{def}}{\Leftrightarrow}$ $M \otimes M$: simple.

Proposition (KKOP)

$\mathcal{D} = \{L_i\}_{i \in I} \subseteq \mathscr{C}_{\mathfrak{g}}$: a strong duality datum associated with C

$\Rightarrow$ $^{\exists!}\mathbb{Z}$-alg. hom. $\Phi_{\mathcal{D}} \colon U_q^-(\mathfrak{g}_C)^\vee_{\mathbb{Z}} \to K(\mathscr{C}_{\mathfrak{g}})$ s.t. $\Phi_{\mathcal{D}}(f_i) = [L_i]$ $(i \in I)$, $\Phi_{\mathcal{D}}(q) = 1$.

Moreover, $\Phi_{\mathcal{D}}$ induces an inj. map from the **upper global basis** $\mathbf{B}^{\mathrm{up}} \subseteq U_q^-(\mathfrak{g}_C)^\vee_{\mathbb{Z}}$ to the isom. classes of simple modules in $\mathscr{C}_{\mathfrak{g}}$:

$$\mathbf{B}^{\mathrm{up}} \ni b \mapsto L_b \in (\text{simples in } \mathscr{C}_{\mathfrak{g}}) \qquad (\textbf{not surjective!})$$

Rem. $\Phi_{\mathcal{D}}$ is defined by the composition of the following two hom.:

- $U_q^-(\mathfrak{g}_C)^\vee_{\mathbb{Z}} \xrightarrow{\sim} K(R^C\text{-gmod})$ (R^C: **quiver Hecke algebra**) [Khovanov–Lauda, Rouquier]
- $K(R^C\text{-gmod}) \to K(\mathscr{C}_{\mathfrak{g}})$, which is induced from the **quantum affine Schur–Weyl duality functor** $R^C\text{-gmod} \to \mathscr{C}_{\mathfrak{g}}$.

Proposition (KKOP)

$\mathcal{D} = \{L_i\}_{i\in I} \subseteq \mathscr{C}_\mathfrak{g}$: a strong duality datum associated with C

$\Rightarrow$ $^{\exists!}\mathbb{Z}$-alg. hom. $\Phi_\mathcal{D} \colon U_q^-(\mathfrak{g}_C)^\vee_\mathbb{Z} \to K(\mathscr{C}_\mathfrak{g})$ s.t. $\Phi_\mathcal{D}(f_i) = [L_i]$ $(i \in I)$, $\Phi_\mathcal{D}(q) = 1$.

Moreover, $\Phi_\mathcal{D}$ induces an inj. map from the **upper global basis** $\mathbf{B}^{\mathrm{up}} \subseteq U_q^-(\mathfrak{g}_C)^\vee_\mathbb{Z}$ to the isom. classes of simple modules in $\mathscr{C}_\mathfrak{g}$:

$$\mathbf{B}^{\mathrm{up}} \ni b \mapsto L_b \in (\text{simples in } \mathscr{C}_\mathfrak{g}) \qquad (\textbf{not surjective!})$$

Rem. $\Phi_\mathcal{D}$ is defined by the composition of the following two hom.:

- $U_q^-(\mathfrak{g}_C)^\vee_\mathbb{Z} \xrightarrow{\sim} K(R^C\text{-gmod})$ (R^C: **quiver Hecke algebra**) [Khovanov–Lauda, Rouquier]
- $K(R^C\text{-gmod}) \to K(\mathscr{C}_\mathfrak{g})$, which is induced from the **quantum affine Schur–Weyl duality functor** $R^C\text{-gmod} \to \mathscr{C}_\mathfrak{g}$.

Proposition (KKOP)

$\mathcal{D} = \{L_i\}_{i \in I} \subseteq \mathscr{C}_{\mathfrak{g}}$: a strong duality datum associated with C

$\Rightarrow$ $^{\exists !}\mathbb{Z}$-alg. hom. $\Phi_{\mathcal{D}} \colon U_q^-(\mathfrak{g}_C)^\vee_{\mathbb{Z}} \to K(\mathscr{C}_{\mathfrak{g}})$ s.t. $\Phi_{\mathcal{D}}(f_i) = [L_i]$ $(i \in I)$, $\Phi_{\mathcal{D}}(q) = 1$.

Moreover, $\Phi_{\mathcal{D}}$ induces an inj. map from the **upper global basis** $\mathbf{B}^{\mathrm{up}} \subseteq U_q^-(\mathfrak{g}_C)^\vee_{\mathbb{Z}}$ to the isom. classes of simple modules in $\mathscr{C}_{\mathfrak{g}}$:

$$\mathbf{B}^{\mathrm{up}} \ni b \mapsto L_b \in (\text{simples in } \mathscr{C}_{\mathfrak{g}}) \qquad (\textbf{not surjective!})$$

Rem. $\Phi_{\mathcal{D}}$ is defined by the composition of the following two hom.:

- $U_q^-(\mathfrak{g}_C)^\vee_{\mathbb{Z}} \xrightarrow{\sim} K(R^C\text{–gmod})$ (R^C: **quiver Hecke algebra**) [Khovanov–Lauda, Rouquier]
- $K(R^C\text{–gmod}) \to K(\mathscr{C}_{\mathfrak{g}})$, which is induced from the **quantum affine Schur–Weyl duality functor** $R^C\text{–gmod} \to \mathscr{C}_{\mathfrak{g}}$.

affine cuspidal modules

$\mathcal{D}$: strong duality datum assoociated with C

$$\rightsquigarrow \Phi_{\mathcal{D}} \colon \begin{array}{ccc} U_q^-(\mathfrak{g}_C)^\vee_{\mathbb{Z}} & \to & K(\mathscr{C}_\mathfrak{g}) \\ \cup & & \cup \\ \mathbf{B}^{\mathrm{up}} & & \{\text{simple mod.}\}/_{\cong} \end{array}$$

Fix a reduced word $\boldsymbol{i} = (i_1, \ldots, i_N)$ of the longest el. w_0 of W_C. For $1 \leq j \leq N$,

$U_q^-(\mathfrak{g}_C)^\vee \ni f_{\beta_j} := T_{i_1} \cdots T_{i_{j-1}}(f_{i_j}) \xrightarrow{\text{normalize}} f^\vee_{\beta_j} \in \mathbf{B}^{\mathrm{up}}$: **dual root vector**

(T_i: Lusztig's braid group action)

Rem. $f^\vee_{\beta_j}$ depends on the choice of $\boldsymbol{i}$.

Definition

Define the **affine cuspidal modules** $\{S_j = S_j^{\mathcal{D},\boldsymbol{i}} \mid j \in \mathbb{Z}\} \subseteq \mathscr{C}_\mathfrak{g}$ as follows:

(i) if $1 \leq j \leq N$, S_j is the image of $f^\vee_{\beta_j}$ under $\Phi_{\mathcal{D}}$, and

(ii) set $S_{j\pm N} = \mathscr{D}^{\mp 1} S_j$ for all $j \in \mathbb{Z}$.

affine cuspidal modules

$\mathcal{D}$: strong duality datum assoociated with C

$$\rightsquigarrow \Phi_{\mathcal{D}} \colon \begin{array}{ccc} U_q^-(\mathfrak{g}_C)^\vee_{\mathbb{Z}} & \to & K(\mathscr{C}_{\mathfrak{g}}) \\ \cup & & \cup \\ \mathbf{B}^{\mathrm{up}} & & \{\text{simple mod.}\}/_{\cong} \end{array}$$

Fix a reduced word $\boldsymbol{i} = (i_1, \ldots, i_N)$ of the longest el. w_0 of W_C. For $1 \leq j \leq N$,

$U_q^-(\mathfrak{g}_C)^\vee \ni f_{\beta_j} := T_{i_1} \cdots T_{i_{j-1}}(f_{i_j}) \xrightarrow{\text{normalize}} f_{\beta_j}^\vee \in \mathbf{B}^{\mathrm{up}}$: **dual root vector**

(T_i: Lusztig's braid group action)

<u>Rem.</u> $f_{\beta_j}^\vee$ depends on the choice of $\boldsymbol{i}$.

Definition

Define the **affine cuspidal modules** $\{S_j = S_j^{\mathcal{D},\boldsymbol{i}} \mid j \in \mathbb{Z}\} \subseteq \mathscr{C}_{\mathfrak{g}}$ as follows:

(i) if $1 \leq j \leq N$, S_j is the image of $f_{\beta_j}^\vee$ under $\Phi_{\mathcal{D}}$, and

(ii) set $S_{j \pm N} = \mathscr{D}^{\mp 1} S_j$ for all $j \in \mathbb{Z}$.

affine cuspidal modules

$\mathcal{D}$: strong duality datum assoociated with C

$$\rightsquigarrow \Phi_{\mathcal{D}}\colon \begin{array}{ccc} U_q^-(\mathfrak{g}_C)_{\mathbb{Z}}^{\vee} & \to & K(\mathscr{C}_{\mathfrak{g}}) \\ \cup & & \cup \\ \mathbf{B}^{\mathrm{up}} & & \{\text{simple mod.}\}/_{\cong} \end{array}$$

Fix a reduced word $\boldsymbol{i} = (i_1, \dots, i_N)$ of the longest el. w_0 of W_C. For $1 \le j \le N$,

$U_q^-(\mathfrak{g}_C)^{\vee} \ni f_{\beta_j} := T_{i_1} \cdots T_{i_{j-1}}(f_{i_j}) \overset{\text{normalize}}{\longrightarrow} f_{\beta_j}^{\vee} \in \mathbf{B}^{\mathrm{up}}$: **dual root vector**

(T_i: Lusztig's braid group action)

Rem. $f_{\beta_j}^{\vee}$ depends on the choice of $\boldsymbol{i}$.

Definition

Define the **affine cuspidal modules** $\{S_j = S_j^{\mathcal{D},\boldsymbol{i}} \mid j \in \mathbb{Z}\} \subseteq \mathscr{C}_{\mathfrak{g}}$ as follows:

(i) if $1 \le j \le N$, S_j is the image of $f_{\beta_j}^{\vee}$ under $\Phi_{\mathcal{D}}$, and

(ii) set $S_{j \pm N} = \mathscr{D}^{\mp 1} S_j$ for all $j \in \mathbb{Z}$.

Examples

Assume $\mathfrak{g} = \widehat{\mathfrak{sl}}_{n+1}$ (type $A_n^{(1)}$).

(1) $\mathcal{D}_0 := \{L(Y_{1,-2j+1}) \mid 1 \leq j \leq n\} \subseteq \mathscr{C}_{\widehat{\mathfrak{sl}}_{n+1}}$: SDD of type A_n

$\boldsymbol{i}_0 := (1, \ldots, n/1, \ldots, n-1/\ldots/1, 2/1)$

$\rightsquigarrow \ldots, S_1 = \Phi_{\mathcal{D}}(f_1) = L(Y_{1,-1}), \; S_2 = \Phi_{\mathcal{D}}(f^{\vee}_{\alpha_1+\alpha_2}) = L(Y_{2,-2}), \ldots,$

$S_{N+1} = \mathscr{D}^{-1}(S_1) = L(Y_{n,-n-2}), \ldots$

$\rightsquigarrow \quad \{S_j^{\mathcal{D}_0, \boldsymbol{i}_0} | j \in \mathbb{Z}\} = \{L(Y_{i,k}) \mid k \equiv i \pmod 2\}$

Aff. cuspidal mod. are a generalization of fund. mod.!

(2) $\mathcal{D} := \{L(Y_{1,2j-1}) \mid 1 \leq j \leq n\}$, $\boldsymbol{i}_0$: as above

$\rightsquigarrow \ldots, S_1 = L(Y_{1,1}), \; S_2 = L(Y_{1,3}Y_{1,1}), \; S_3 = L(Y_{1,5}Y_{1,3}Y_{1,1}), \ldots$

Examples

Assume $\mathfrak{g} = \widehat{\mathfrak{sl}}_{n+1}$ (type $A_n^{(1)}$).

(1) $\mathcal{D}_0 := \{L(Y_{1,-2j+1}) \mid 1 \leq j \leq n\} \subseteq \mathscr{C}_{\widehat{\mathfrak{sl}}_{n+1}}$: SDD of type A_n

$\boldsymbol{i}_0 := (1, \ldots, n/1, \ldots, n-1/\ldots/1, 2/1)$

$\rightsquigarrow \ldots,\ S_1 = \Phi_{\mathcal{D}}(f_1) = L(Y_{1,-1}),\ S_2 = \Phi_{\mathcal{D}}(f^{\vee}_{\alpha_1+\alpha_2}) = L(Y_{2,-2}),\ \ldots,$

$S_{N+1} = \mathscr{D}^{-1}(S_1) = L(Y_{n,-n-2}), \ldots$

$\rightsquigarrow \quad \{S_j^{\mathcal{D}_0, \boldsymbol{i}_0} | j \in \mathbb{Z}\} = \{L(Y_{i,k}) \mid k \equiv i \pmod 2\}$

Aff. cuspidal mod. are a generalization of fund. mod.!

(2) $\mathcal{D} := \{L(Y_{1,2j-1}) \mid 1 \leq j \leq n\}$, $\boldsymbol{i}_0$: as above

$\rightsquigarrow \ldots,\ S_1 = L(Y_{1,1}),\ S_2 = L(Y_{1,3}Y_{1,1}),\ S_3 = L(Y_{1,5}Y_{1,3}Y_{1,1}),\ \ldots$

Main Theorem

Mukhin–Young's extended T-systems are generalized to a general strong duality data of type A.

1. Mukhin–Young's extended T-systems (what we generalize)
2. Strong duality data and affine cuspidal modules [KKOP]
3. Main Theorem
4. Proof (relations between the extended T-systems and Kashiwara crystals)

Rem.

We generalized both types ($A_n^{(1)}$ and $B_n^{(1)}$) of Mukhin-Young's extended T-systems.

For simplicity, we mainly introduce the results in type $A_n^{(1)}$.

Prior work for T-systems by [KKOP]

$\mathcal{D} \subseteq \mathscr{C}_{\mathfrak{g}}$: SDD of arbitrary type, $\boldsymbol{i}$: arbitrary reduced word of the longest el. w_0,

$\rightsquigarrow \{S_j = S_j^{\mathcal{D},\boldsymbol{i}} \mid j \in \mathbb{Z}\}$: the associated affine cuspidal modules

A family of simple modules $\{M_i[a,b] \mid i \in I,\ a < b\}$ was defined, where

$M_i[a,b] := \mathrm{hd}(S_{r_1} \otimes \cdots \otimes S_{r_p})$ with a suitable seq. $r_1 < r_2 < \cdots < r_p$ of integers.

Fact In the case where $\{S_j\}$ are fundamental modules,

$$M_i[a,b] = \mathrm{hd}\big(L(Y_{i,r+2}) \otimes \cdots \otimes L(Y_{i,r+2k})\big) \cong L\Big(\prod_{k=1}^{p} Y_{i,r+2k}\Big)\text{: KR modules.}$$

Theorem (KKOP)

$$0 \to \bigotimes_{j;c_{ij}=-1} M_j[a(j)^+, b(j)^-] \to M_i[a^+,b] \otimes M_i[a,b^-] \to M_i[a,b] \otimes M_i[a^+,b^-] \to 0$$

- The first and the third terms are both simple.

$\rightsquigarrow$ T-systems as the special cases of KR modules

Prior work for T-systems by [KKOP]

$\mathcal{D} \subseteq \mathscr{C}_{\mathfrak{g}}$: SDD of arbitrary type, $\boldsymbol{i}$: arbitrary reduced word of the longest el. w_0,

$\rightsquigarrow \{S_j = S_j^{\mathcal{D},\boldsymbol{i}} \mid j \in \mathbb{Z}\}$: the associated affine cuspidal modules

A family of simple modules $\{M_i[a,b] \mid i \in I,\ a < b\}$ was defined, where

$M_i[a,b] := \mathrm{hd}(S_{r_1} \otimes \cdots \otimes S_{r_p})$ with a suitable seq. $r_1 < r_2 < \cdots < r_p$ of integers.

Fact In the case where $\{S_j\}$ are fundamental modules,

$$M_i[a,b] = \mathrm{hd}\big(L(Y_{i,r+2}) \otimes \cdots \otimes L(Y_{i,r+2k})\big) \cong L\Big(\prod_{k=1}^{p} Y_{i,r+2k}\Big): \text{ KR modules.}$$

Theorem (KKOP)

$$0 \to \bigotimes_{j;c_{ij}=-1} M_j[a(j)^+, b(j)^-] \to M_i[a^+, b] \otimes M_i[a, b^-] \to M_i[a,b] \otimes M_i[a^+, b^-] \to 0$$

- The first and the third terms are both simple.

$\rightsquigarrow$ T-systems as the special cases of KR modules

Prior work for T-systems by [KKOP]

$\mathcal{D} \subseteq \mathscr{C}_{\mathfrak{g}}$: SDD of arbitrary type, $\boldsymbol{i}$: arbitrary reduced word of the longest el. w_0,

$\rightsquigarrow \{S_j = S_j^{\mathcal{D},\boldsymbol{i}} \mid j \in \mathbb{Z}\}$: the associated affine cuspidal modules

A family of simple modules $\{M_i[a,b] \mid i \in I,\ a < b\}$ was defined, where

$M_i[a,b] := \mathrm{hd}(S_{r_1} \otimes \cdots \otimes S_{r_p})$ with a suitable seq. $r_1 < r_2 < \cdots < r_p$ of integers.

<u>Fact</u> In the case where $\{S_j\}$ are fundamental modules,

$$M_i[a,b] = \mathrm{hd}\big(L(Y_{i,r+2}) \otimes \cdots \otimes L(Y_{i,r+2k})\big) \cong L\Big(\prod_{k=1}^{p} Y_{i,r+2k}\Big)\text{: KR modules.}$$

Theorem (KKOP)

$$0 \to \bigotimes_{j;c_{ij}=-1} M_j[a(j)^+, b(j)^-] \to M_i[a^+,b] \otimes M_i[a,b^-] \to M_i[a,b] \otimes M_i[a^+,b^-] \to 0$$

- The first and the third terms are both simple.

$\rightsquigarrow$ T-systems as the special cases of KR modules

Main Theorem

Setting $\mathcal{D} \in \mathscr{C}_{\mathfrak{g}}$: a strong duality datum of type A_n,

We consider **only the special** reduced word $\boldsymbol{i}_0 := (1, \dots, n/ \dots /1, 2/1)$

$\rightsquigarrow$ affine cuspidal modules $S_j^{\mathcal{D}, \boldsymbol{i}_0}$ $(j \in \mathbb{Z})$

Set $J_n = \{(i,k) \in [1,n] \times \mathbb{Z} \mid i \equiv k \bmod 2\}$, and define

$$S_{i,k} := S_j^{\mathcal{D}, \boldsymbol{i}_0} \quad \text{for } (i,k) \in J_n,$$

where $j \in \mathbb{Z}$ is s.t. $S_j^{\mathcal{D}_0, \boldsymbol{i}_0} = L(Y_{i,k})$.

Here $\mathcal{D}_0$ is the special SDD s.t. $\{S_j^{\mathcal{D}_0, \boldsymbol{i}_0} \mid j \in \mathbb{Z}\} = \{L(Y_{i,k}) \mid (i,k) \in J_n\}$ appearing in the previous example.

Definition (Snake module associated with $\mathcal{D}$)

$\mathbb{S}^{\mathcal{D}}(\xi) := \mathrm{hd}(S_{i_1,k_1} \otimes \cdots \otimes S_{i_p,k_p})$ for a snake $\xi = ((i_1,k_1), \dots, (i_p.k_p)) \in J_n^p$.

Main Theorem

Setting $\mathcal{D} \in \mathscr{C}_{\mathfrak{g}}$: a strong duality datum of type A_n,

We consider **only the special** reduced word $\boldsymbol{i}_0 := (1, \dots, n/\dots/1, 2/1)$

$\rightsquigarrow$ affine cuspidal modules $S_j^{\mathcal{D},\boldsymbol{i}_0}$ ($j \in \mathbb{Z}$)

Set $J_n = \{(i,k) \in [1,n] \times \mathbb{Z} \mid i \equiv k \bmod 2\}$, and define

$$S_{i,k} := S_j^{\mathcal{D},\boldsymbol{i}_0} \quad \text{for } (i,k) \in J_n,$$

where $j \in \mathbb{Z}$ is s.t. $S_j^{\mathcal{D}_0,\boldsymbol{i}_0} = L(Y_{i,k})$.

Here $\mathcal{D}_0$ is the special SDD s.t. $\{S_j^{\mathcal{D}_0,\boldsymbol{i}_0} \mid j \in \mathbb{Z}\} = \{L(Y_{i,k}) \mid (i,k) \in J_n\}$ appearing in the previous example.

Definition (Snake module associated with $\mathcal{D}$)

$\mathbb{S}^{\mathcal{D}}(\xi) := \mathrm{hd}(S_{i_1,k_1} \otimes \cdots \otimes S_{i_p,k_p})$ for a snake $\xi = ((i_1,k_1),\dots,(i_p.k_p)) \in J_n^p$.

Main Theorem

Setting $\mathcal{D} \in \mathscr{C}_{\mathfrak{g}}$: a strong duality datum of type A_n,

We consider **only the special** reduced word $\boldsymbol{i}_0 := (1, \ldots, n/ \ldots /1, 2/1)$

$\rightsquigarrow$ affine cuspidal modules $S_j^{\mathcal{D}, \boldsymbol{i}_0}$ $(j \in \mathbb{Z})$

Set $J_n = \{(i,k) \in [1,n] \times \mathbb{Z} \mid i \equiv k \bmod 2\}$, and define

$$S_{i,k} := S_j^{\mathcal{D}, \boldsymbol{i}_0} \quad \text{for } (i,k) \in J_n,$$

where $j \in \mathbb{Z}$ is s.t. $S_j^{\mathcal{D}_0, \boldsymbol{i}_0} = L(Y_{i,k})$.

Here $\mathcal{D}_0$ is the special SDD s.t. $\{S_j^{\mathcal{D}_0, \boldsymbol{i}_0} \mid j \in \mathbb{Z}\} = \{L(Y_{i,k}) \mid (i,k) \in J_n\}$ appearing in the previous example.

Definition (Snake module associated with $\mathcal{D}$)

$\mathbb{S}^{\mathcal{D}}(\boldsymbol{\xi}) := \mathrm{hd}(S_{i_1,k_1} \otimes \cdots \otimes S_{i_p,k_p})$ for a snake $\boldsymbol{\xi} = \big((i_1,k_1), \ldots, (i_p.k_p)\big) \in J_n^p$.

Definition (Snake module associated with $\mathcal{D}$)

$\mathbb{S}^{\mathcal{D}}(\boldsymbol{\xi}) := \mathrm{hd}(S_{i_1,k_1} \otimes \cdots \otimes S_{i_p,k_p})$ for a snake $\boldsymbol{\xi} = \big((i_1,k_1),\ldots,(i_p.k_p)\big) \in J_n^p$.

Rem. When $\mathcal{D} = \mathcal{D}_0$,

$$\mathbb{S}^{\mathcal{D}_0}(\boldsymbol{\xi}) = \mathrm{hd}\big(L(Y_{i_1,k_1}) \otimes \cdots \otimes L(Y_{i_p,k_p})\big) \cong L\Big(\prod_{r=1}^{p} Y_{i_r,k_r}\Big) = L(\boldsymbol{\xi})$$ (MY snake mod.)

Theorem (N)

For a prime snake $\xi \in J_n^p$,

$$0 \to \mathbb{S}^{\mathcal{D}}(\xi_{\mathrm{H}}) \otimes \mathbb{S}^{\mathcal{D}}(\xi_{\mathrm{L}}) \to \mathbb{S}^{\mathcal{D}}(\xi_{[1,p-1]}) \otimes \mathbb{S}^{\mathcal{D}}(\xi_{[2,p]}) \to \mathbb{S}^{\mathcal{D}}(\xi) \otimes \mathbb{S}^{\mathcal{D}}(\xi_{[2,p-1]}) \to 0$$

- The first and the third terms are simple.

$\rightsquigarrow [L(\xi_{[1,p-1]}) \otimes L(\xi_{[2,p]})] = [L(\xi) \otimes L(\xi_{[2,p-1]})] + [L(\xi_{\mathrm{H}}) \otimes L(\xi_{\mathrm{L}})]$ when $\mathcal{D} = \mathcal{D}_0$.

Definition (Snake module associated with $\mathcal{D}$)

$\mathbb{S}^{\mathcal{D}}(\boldsymbol{\xi}) := \mathrm{hd}(S_{i_1,k_1} \otimes \cdots \otimes S_{i_p,k_p})$ for a snake $\boldsymbol{\xi} = \big((i_1,k_1),\ldots,(i_p.k_p)\big) \in J_n^p$.

Rem. When $\mathcal{D} = \mathcal{D}_0$,

$$\mathbb{S}^{\mathcal{D}_0}(\boldsymbol{\xi}) = \mathrm{hd}\big(L(Y_{i_1,k_1}) \otimes \cdots \otimes L(Y_{i_p,k_p})\big) \simeq L\Big(\prod_{r=1}^{p} Y_{i_r,k_r}\Big) = L(\boldsymbol{\xi}) \text{ (MY snake mod.)}$$

Theorem (N)

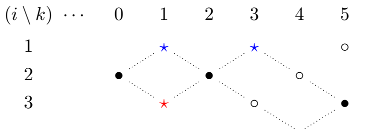
For a prime snake $\boldsymbol{\xi} \in J_n^p$,

$$0 \to \mathbb{S}^{\mathcal{D}}(\boldsymbol{\xi}_{\mathrm{H}}) \otimes \mathbb{S}^{\mathcal{D}}(\boldsymbol{\xi}_{\mathrm{L}}) \to \mathbb{S}^{\mathcal{D}}(\boldsymbol{\xi}_{[1,p-1]}) \otimes \mathbb{S}^{\mathcal{D}}(\boldsymbol{\xi}_{[2,p]}) \to \mathbb{S}^{\mathcal{D}}(\boldsymbol{\xi}) \otimes \mathbb{S}^{\mathcal{D}}(\boldsymbol{\xi}_{[2,p-1]}) \to 0$$

- The first and the third terms are simple.

$\rightsquigarrow$ $\big[L(\boldsymbol{\xi}_{[1,p-1]}) \otimes L(\boldsymbol{\xi}_{[2,p]})\big] = \big[L(\boldsymbol{\xi}) \otimes L(\boldsymbol{\xi}_{[2,p-1]})\big] + \big[L(\boldsymbol{\xi}_{\mathrm{H}}) \otimes L(\boldsymbol{\xi}_{\mathrm{L}})\big]$ when $\mathcal{D} = \mathcal{D}_0$.

Ex. type A_3, $\boldsymbol{i}_0 = (1,2,3,1,2,1)$



$$\rightsquigarrow \quad 0 \to \mathbb{S}^{\mathcal{D}}\big((1,3),(1,1)\big) \otimes \mathbb{S}^{\mathcal{D}}\big((3,1)\big) \to \mathbb{S}^{\mathcal{D}}\big((3,5),(2,2)\big) \otimes \mathbb{S}^{\mathcal{D}}\big((2,2),(2,0)\big) \to \mathbb{S}^{\mathcal{D}}\big((3,5),(2,2),(2,0)\big) \otimes \mathbb{S}^{\mathcal{D}}\big((2,2)\big) \to 0$$

(i) $L_i = L(Y_{1,-2i+1}) \rightsquigarrow S_{i,k} = L(Y_{i,k}) \rightsquigarrow \mathbb{S}^{\mathcal{D}}\big((i_1,k_1),\cdots,(i_p,k_p)\big) = L(\prod_{r=1}^{p} Y_{i_r,k_r})$

$$L(Y_{1,3}Y_{1,1}) \otimes L(Y_{3,1}) \to L(Y_{3,5}Y_{2,2}) \otimes Y(Y_{2,2}Y_{2,0}) \to L(Y_{3,5}Y_{2,2}Y_{2,0}) \otimes L(Y_{2,2})$$

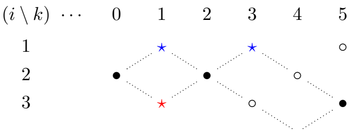
(ii) $L_i = L(Y_{1,2i-7}) \rightsquigarrow S_{3,5} = L(Y_{1,7}Y_{1,5}Y_{1,3}),\ S_{2,2} = L(Y_{3,1}Y_{3,-1}),$

$S_{2,0} = L(Y_{3,3}Y_{3,1}),\ S_{1,3} = L(Y_{1,7}), \ldots$, etc.

$$\rightsquigarrow \mathbb{S}^{\mathcal{D}}\big((3,5),(2,2),(2,0)\big) = \mathrm{hd}\big(L(Y_{1,7}Y_{1,5}Y_{1,3}) \otimes L(Y_{3,1}Y_{3,-1}) \otimes L(Y_{3,3}Y_{3,1})\big)$$
$$= L(Y_{1,7}Y_{1,5}Y_{1,3}Y_{2,2}Y_{2,0}), \text{ etc.}$$

$$L(Y_{1,7}Y_{3,3}Y_{3,1}Y_{3,-1}) \otimes L(Y_{1,-3}) \to L(Y_{1,7}Y_{1,5}Y_{1,3}Y_{3,1}Y_{3,-1}) \otimes L(Y_{2,2}Y_{2,0})$$
$$\to L(Y_{1,7}Y_{1,5}Y_{1,3}Y_{2,2}Y_{2,0}) \otimes L(Y_{3,1}Y_{3-1})$$

Ex. type A_3, $\boldsymbol{i}_0 = (1,2,3,1,2,1)$



$\rightsquigarrow$

$$0 \to \mathbb{S}^{\mathcal{D}}\big((1,3),(1,1)\big) \otimes \mathbb{S}^{\mathcal{D}}\big((3,1)\big)$$
$$\to \mathbb{S}^{\mathcal{D}}\big((3,5),(2,2)\big) \otimes \mathbb{S}^{\mathcal{D}}\big((2,2),(2,0)\big)$$
$$\to \mathbb{S}^{\mathcal{D}}\big((3,5),(2,2),(2,0)\big) \otimes \mathbb{S}^{\mathcal{D}}\big((2,2)\big) \to 0$$

(i) $L_i = L(Y_{1,-2i+1}) \rightsquigarrow S_{i,k} = L(Y_{i,k}) \rightsquigarrow \mathbb{S}^{\mathcal{D}}\big((i_1,k_1),\cdots,(i_p,k_p)\big) = L(\prod_{r=1}^{p} Y_{i_r,k_r})$

$$L(Y_{1,3}Y_{1,1}) \otimes L(Y_{3,1}) \to L(Y_{3,5}Y_{2,2}) \otimes Y(Y_{2,2}Y_{2,0}) \to L(Y_{3,5}Y_{2,2}Y_{2,0}) \otimes L(Y_{2,2})$$

(ii) $L_i = L(Y_{1,2i-7}) \rightsquigarrow S_{3,5} = L(Y_{1,7}Y_{1,5}Y_{1,3})$, $S_{2,2} = L(Y_{3,1}Y_{3,-1})$,

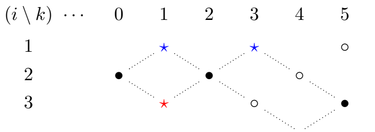
$S_{2,0} = L(Y_{3,3}Y_{3,1})$, $S_{1,3} = L(Y_{1,7})$, ..., etc.

$\rightsquigarrow \mathbb{S}^{\mathcal{D}}\big((3,5),(2,2),(2,0)\big) = \mathrm{hd}\big(L(Y_{1,7}Y_{1,5}Y_{1,3}) \otimes L(Y_{3,1}Y_{3,-1}) \otimes L(Y_{3,3}Y_{3,1})\big)$

$= L(Y_{1,7}Y_{1,5}Y_{1,3}Y_{2,2}Y_{2,0})$, etc.

$$L(Y_{1,7}Y_{3,3}Y_{3,1}Y_{3,-1}) \otimes L(Y_{1,-3}) \to L(Y_{1,7}Y_{1,5}Y_{1,3}Y_{3,1}Y_{3,-1}) \otimes L(Y_{2,2}Y_{2,0})$$
$$\to L(Y_{1,7}Y_{1,5}Y_{1,3}Y_{2,2}Y_{2,0}) \otimes L(Y_{3,1}Y_{3-1})$$

Ex. type A_3, $\boldsymbol{i}_0 = (1,2,3,1,2,1)$



$\rightsquigarrow$

$$0 \to \mathbb{S}^{\mathcal{D}}\big((1,3),(1,1)\big) \otimes \mathbb{S}^{\mathcal{D}}\big((3,1)\big) \to \mathbb{S}^{\mathcal{D}}\big((3,5),(2,2)\big) \otimes \mathbb{S}^{\mathcal{D}}\big((2,2),(2,0)\big) \to \mathbb{S}^{\mathcal{D}}\big((3,5),(2,2),(2,0)\big) \otimes \mathbb{S}^{\mathcal{D}}\big((2,2)\big) \to 0$$

(i) $L_i = L(Y_{1,-2i+1}) \rightsquigarrow S_{i,k} = L(Y_{i,k}) \rightsquigarrow \mathbb{S}^{\mathcal{D}}\big((i_1,k_1),\cdots,(i_p,k_p)\big) = L(\prod_{r=1}^{p} Y_{i_r,k_r})$

$$L(Y_{1,3}Y_{1,1}) \otimes L(Y_{3,1}) \to L(Y_{3,5}Y_{2,2}) \otimes Y(Y_{2,2}Y_{2,0}) \to L(Y_{3,5}Y_{2,2}Y_{2,0}) \otimes L(Y_{2,2})$$

(ii) $L_i = L(Y_{1,2i-7}) \rightsquigarrow S_{3,5} = L(Y_{1,7}Y_{1,5}Y_{1,3})$, $S_{2,2} = L(Y_{3,1}Y_{3,-1})$,
$S_{2,0} = L(Y_{3,3}Y_{3,1})$, $S_{1,3} = L(Y_{1,7})$, …, etc.

$$\rightsquigarrow \mathbb{S}^{\mathcal{D}}\big((3,5),(2,2),(2,0)\big) = \mathrm{hd}\big(L(Y_{1,7}Y_{1,5}Y_{1,3}) \otimes L(Y_{3,1}Y_{3,-1}) \otimes L(Y_{3,3}Y_{3,1})\big)$$
$$= L(Y_{1,7}Y_{1,5}Y_{1,3}Y_{2,2}Y_{2,0}), \text{ etc.}$$

$$L(Y_{1,7}Y_{3,3}Y_{3,1}Y_{3,-1}) \otimes L(Y_{1,-3}) \to L(Y_{1,7}Y_{1,5}Y_{1,3}Y_{3,1}Y_{3,-1}) \otimes L(Y_{2,2}Y_{2,0})$$
$$\to L(Y_{1,7}Y_{1,5}Y_{1,3}Y_{2,2}Y_{2,0}) \otimes L(Y_{3,1}Y_{3-1})$$

Main Theorem

Mukhin–Young's extended T-systems are generalized to a general strong duality data of type A.

1. Mukhin–Young's extended T-systems (what we generalize)
2. Strong duality data and affine cuspidal modules [KKOP]
3. Main Theorem
4. Proof (relations between the extended T-systems and Kashiwara crystals)

Key Lemma

$\mathcal{D}, \boldsymbol{i}$: arbitrary $\rightsquigarrow \{S_j = S_j^{\mathcal{D},\boldsymbol{i}} \mid j \in \mathbb{Z}\}$: affine cuspidal modules

Lemma (KKOP, N)

For a sequence $\boldsymbol{k} = (k_1 < k_2 < \cdots < k_p) \in \mathbb{Z}^p$, denote by

$$\mathbb{S}_{\boldsymbol{k}}[s,t] = \mathrm{hd}(S_{k_s} \otimes S_{k_{s+1}} \otimes \cdots \otimes S_{k_t}) \quad (1 \le s \le t \le p).$$

Assume that

(i) $\mathfrak{d}\big(S_{k_s}, \mathbb{S}_{\boldsymbol{k}}[s+1,t]\big) = 1$ for all $1 \le s < t \le p$,

(ii) $\mathfrak{d}\big(\mathbb{S}_{\boldsymbol{k}}[s,t-1], S_{k_t}\big) = 1$ for all $1 \le s < t \le p$.

Then we have

$$0 \to \mathrm{hd}\Big(\bigotimes_{r=1}^{p-1} \mathrm{hd}(S_{k_{r+1}} \otimes S_{k_r})\Big) \to \mathbb{S}_{\boldsymbol{k}}[1,p-1] \otimes \mathbb{S}_{\boldsymbol{k}}[2,p] \to \mathbb{S}_{\boldsymbol{k}}[1,p] \otimes \mathbb{S}_{\boldsymbol{k}}[2,p-1] \to 0.$$

Moreover, the first and the third terms are both simple.

Q. How to calculate the values of $\mathfrak{d}$?

$$\Phi_{\mathcal{D}}: \quad U_q^-(\mathfrak{g}_C)^\vee_{\mathbb{Z}} \quad \to \quad K(\mathscr{C}_{\mathfrak{g}})$$

$$\cup \qquad\qquad\qquad \cup$$

$$\{(f^\vee_{\beta_1})^{a_1}\cdots(f^\vee_{\beta_N})^{a_N} \mid \boldsymbol{a}=(a_1,\dots,a_N)\in\mathbb{Z}^N_{\geq 0}\} \qquad \{S_1^{\otimes a_1}\otimes\cdots\otimes S_N^{\otimes a_N} \mid \boldsymbol{a}\in\mathbb{Z}^N_{\geq 0}\}$$

dual PBW basis (depending on $\boldsymbol{i}$) $\qquad \updownarrow$

$$\mathbf{B}^{\mathrm{up}} = \{b^{\boldsymbol{i}}(\boldsymbol{a}) \mid \boldsymbol{a}\in\mathbb{Z}^N_{\geq 0}\} \qquad \{\mathrm{hd}(S_1^{\otimes a_1}\otimes\cdots\otimes S_N^{\otimes a_N}) \mid \boldsymbol{a}\in\mathbb{Z}^N_{\geq 0}\}$$

Rem. $\mathbf{B}^{\mathrm{up}}$ has a Kashiwara (bi-)crystal structure.

Proposition (KKOP)

1. $\mathfrak{d}(\mathscr{D}L_i, \mathrm{hd}(S_1^{\otimes a_1}\otimes\cdots\otimes S_N^{\otimes a_N})) = \varepsilon_i(b^{\boldsymbol{i}}(\boldsymbol{a}))$,
2. $\mathfrak{d}(\mathrm{hd}(S_1^{\otimes a_1}\otimes\cdots\otimes S_N^{\otimes a_N}), \mathscr{D}^{-1}L_i) = \varepsilon_i^*(b^{\boldsymbol{i}}(\boldsymbol{a}))$,

where $\varepsilon_i(b) = \max\{r\in\mathbb{Z}_{\geq 0} \mid \tilde{e}_i^r(b)\neq 0\}$, and ε_i^* is defined similarly for $\tilde{e}_i^*$.

$$\begin{array}{ccccc} \Phi_{\mathcal{D}}: & U_q^-(\mathfrak{g}_C)^\vee_{\mathbb{Z}} & \to & K(\mathscr{C}_{\mathfrak{g}}) \\ & \cup & & \cup \\ & \{(f^\vee_{\beta_1})^{a_1}\cdots(f^\vee_{\beta_N})^{a_N} \mid \boldsymbol{a}=(a_1,\dots,a_N)\in\mathbb{Z}^N_{\geq 0}\} & & \{S_1^{\otimes a_1}\otimes\cdots\otimes S_N^{\otimes a_N} \mid \boldsymbol{a}\in\mathbb{Z}^N_{\geq 0}\} \\ & \updownarrow \text{ dual PBW basis (depending on } \boldsymbol{i}) & & \updownarrow \\ & \mathbf{B}^{\mathrm{up}} = \{b^{\boldsymbol{i}}(\boldsymbol{a}) \mid \boldsymbol{a}\in\mathbb{Z}^N_{\geq 0}\} & & \{\mathrm{hd}(S_1^{\otimes a_1}\otimes\cdots\otimes S_N^{\otimes a_N}) \mid \boldsymbol{a}\in\mathbb{Z}^N_{\geq 0}\} \end{array}$$

Rem. $\mathbf{B}^{\mathrm{up}}$ has a Kashiwara (bi-)crystal structure.

Proposition (KKOP)

1. $\mathfrak{d}(\mathscr{D}L_i, \mathrm{hd}(S_1^{\otimes a_1}\otimes\cdots\otimes S_N^{\otimes a_N})) = \varepsilon_i(b^{\boldsymbol{i}}(\boldsymbol{a}))$,
2. $\mathfrak{d}(\mathrm{hd}(S_1^{\otimes a_1}\otimes\cdots\otimes S_N^{\otimes a_N}), \mathscr{D}^{-1}L_i) = \varepsilon_i^*(b^{\boldsymbol{i}}(\boldsymbol{a}))$,

where $\varepsilon_i(b) = \max\{r\in\mathbb{Z}_{\geq 0} \mid \tilde{e}_i^r(b)\neq 0\}$, and ε_i^* is defined similarly for $\tilde{e}_i^*$.

$$
\begin{array}{ccc}
\Phi_{\mathcal{D}}\colon \quad U_q^-(\mathfrak{g}_C)^\vee_{\mathbb{Z}} & \to & K(\mathscr{C}_{\mathfrak{g}}) \\
\cup & & \cup \\
\{(f^\vee_{\beta_1})^{a_1}\cdots(f^\vee_{\beta_N})^{a_N} \mid \boldsymbol{a}=(a_1,\dots,a_N)\in\mathbb{Z}^N_{\geq 0}\} & & \{S_1^{\otimes a_1}\otimes\cdots\otimes S_N^{\otimes a_N} \mid \boldsymbol{a}\in\mathbb{Z}^N_{\geq 0}\} \\
\updownarrow \ \text{dual PBW basis (depending on } \boldsymbol{i}) & & \updownarrow \\
\mathbf{B}^{\mathrm{up}} = \{b^{\boldsymbol{i}}(\boldsymbol{a}) \mid \boldsymbol{a}\in\mathbb{Z}^N_{\geq 0}\} & & \{\mathrm{hd}(S_1^{\otimes a_1}\otimes\cdots\otimes S_N^{\otimes a_N}) \mid \boldsymbol{a}\in\mathbb{Z}^N_{\geq 0}\}
\end{array}
$$

Rem. $\mathbf{B}^{\mathrm{up}}$ has a Kashiwara (bi-)crystal structure.

Proposition (KKOP)

1. $\mathfrak{d}\big(\mathscr{D}L_i, \mathrm{hd}(S_1^{\otimes a_1}\otimes\cdots\otimes S_N^{\otimes a_N})\big) = \varepsilon_i\big(b^{\boldsymbol{i}}(\boldsymbol{a})\big)$,
2. $\mathfrak{d}\big(\mathrm{hd}(S_1^{\otimes a_1}\otimes\cdots\otimes S_N^{\otimes a_N}), \mathscr{D}^{-1}L_i\big) = \varepsilon_i^*\big(b^{\boldsymbol{i}}(\boldsymbol{a})\big)$,

where $\varepsilon_i(b) = \max\{r\in\mathbb{Z}_{\geq 0} \mid \tilde{e}_i^r(b)\neq 0\}$, and ε_i^* is defined similarly for $\tilde{e}_i^*$.

We return to the setting of our main theorem ($\mathcal{D}$: type A, $\boldsymbol{i} = \boldsymbol{i}_0$).

Lemma

If $\mathbb{S} := \mathrm{hd}(S_k^{\otimes a_k} \otimes S_{k+1}^{\otimes a_{k+1}} \otimes \cdots \otimes S_\ell^{\otimes a_\ell})$ is a prime snake module, then we have

$$\mathfrak{d}\big(S_s, \mathrm{hd}(S_{s+1}^{\otimes a_{s+1}} \otimes \cdots \otimes S_t^{\otimes a_t})\big) = \mathfrak{d}\big(\mathrm{hd}(S_s^{\otimes a_s} \otimes \cdots \otimes S_{t-1}^{\otimes a_{t-1}}), S_t\big) = 1$$

for all $1 \le s < t \le p$.

proof) By the previous proposition, the proof is reduced to showing $\varepsilon_i(b) = 1$ or $\varepsilon_i^*(b) = 1$ for certain elements $b \in \mathbf{B}^{\mathrm{up}}$ and $i \in [1, n]$. This is achieved by using the **Reineke's algorithm**, which gives a useful combinatorial algorithm to calculate these values.

Comments on the extended T-systems of type $B_n^{(1)}$

Mukhin–Young also proved the extended T-systems for QAA of type $B_n^{(1)}$:

$$\left[L(\boldsymbol{\xi}_{[1,p-1]}) \otimes L(\boldsymbol{\xi}_{[2,p]})\right] = \left[L(\boldsymbol{\xi}) \otimes L(\boldsymbol{\xi}_{[2,p-1]})\right] + \left[L(\boldsymbol{\xi}_{\mathrm{H}}) \otimes L(\boldsymbol{\xi}_{\mathrm{L}})\right]$$

We can also generalize this to a general strong duality datum setting:

$$0 \to \mathbb{S}^{\mathcal{D}}(\boldsymbol{\xi}_{\mathrm{H}}) \otimes \mathbb{S}^{\mathcal{D}}(\boldsymbol{\xi}_{\mathrm{L}}) \to \mathbb{S}^{\mathcal{D}}(\boldsymbol{\xi}_{[1,p-1]}) \otimes \mathbb{S}^{\mathcal{D}}(\boldsymbol{\xi}_{[2,p]}) \to \mathbb{S}^{\mathcal{D}}(\boldsymbol{\xi}) \otimes \mathbb{S}^{\mathcal{D}}(\boldsymbol{\xi}_{[2,p-1]}) \to 0.$$

Here $\mathbb{S}^{\mathcal{D}}(\boldsymbol{\xi})$ are defined from an arbitrary strong duality datum of **type A_{2n-1}**, and a **different reduced word**

$$\boldsymbol{i}_1 = (1, \dots, 2n-1/n/1, \dots, 2n-2/n/\dots/1, \dots, n/1, \dots, n-2/\dots/12/1).$$

Thank you for your concentration!

Comments on the extended T-systems of type $B_n^{(1)}$

Mukhin–Young also proved the extended T-systems for QAA of type $B_n^{(1)}$:

$$[L(\boldsymbol{\xi}_{[1,p-1]}) \otimes L(\boldsymbol{\xi}_{[2,p]})] = [L(\boldsymbol{\xi}) \otimes L(\boldsymbol{\xi}_{[2,p-1]})] + [L(\boldsymbol{\xi}_{\mathrm{H}}) \otimes L(\boldsymbol{\xi}_{\mathrm{L}})]$$

We can also generalize this to a general strong duality datum setting:

$$0 \to \mathbb{S}^{\mathcal{D}}(\boldsymbol{\xi}_{\mathrm{H}}) \otimes \mathbb{S}^{\mathcal{D}}(\boldsymbol{\xi}_{\mathrm{L}}) \to \mathbb{S}^{\mathcal{D}}(\boldsymbol{\xi}_{[1,p-1]}) \otimes \mathbb{S}^{\mathcal{D}}(\boldsymbol{\xi}_{[2,p]}) \to \mathbb{S}^{\mathcal{D}}(\boldsymbol{\xi}) \otimes \mathbb{S}^{\mathcal{D}}(\boldsymbol{\xi}_{[2,p-1]}) \to 0.$$

Here $\mathbb{S}^{\mathcal{D}}(\boldsymbol{\xi})$ are defined from an arbitrary strong duality datum of **type A_{2n-1}**, and a **different reduced word**

$$\boldsymbol{i}_1 = (1, \dots, 2n-1/n/1, \dots, 2n-2/n/\dots/1, \dots, n/1, \dots, n-2/\dots/12/1).$$

Thank you for your concentration!