# Two-component soliton systems and the Painlevé equations

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**Abstract**

An extension of the two-component KP hierarchy by using a time dependent spectral parameter is proposed. The linear system whose coefficients are $2 \times 2$ matrices is obtained from the hierarchy through a reduction procedure. The Lax pair of the sixth Painlevé equation is obtained from this linear system. A unified approach to treat the other Painlevé equations from the usual two-component KP hierarchy is also presented.

## 1 Introduction

In this paper, we deal with the Painlevé equations, the soliton systems, and the relations between them. In the beginning of this section, we explain the historical review of them. We first mention the Painlevé equations. We pay attention to the connection between them and the monodromy preserving deformation of linear differential equations. We touch on the soliton systems next. The Kadomtsev-Petviashvili (KP) hierarchy arises from the isospectral deformation of the eigenvalue problem. Moreover, we refer to the relations between the soliton systems and the Painlevé equations. They are backed by the connections between the isospectral deformation and the monodromy preserving deformation. We describe the purposes in this paper based on these facts. The aim of this paper is to clarify more definitely these connections. We construct an extension of the two-component KP hierarchy by using a time dependent spectral parameter. We prove the relation between this hierarchy and the sixth Painlevé equation. We also show the relation between the usual two-component KP hierarchy and the other Painlevé equations.

## 1.1 Painlevé equations

P. Painlevé studied on second order ordinary differential equations of the form

$$\frac{d^2y}{dt^2} = F\left(t, y, \frac{dy}{dt}\right) \tag{1.1}$$

where $F$ is analytic in $t$ and rational in $y$ and $dy/dt$. He sought the equations whose only movable singularities are poles. This characteristic is known as the Painlevé property. He found the six new equations known as the Painlevé equations ([28]):

$$\mathrm{P_I}: \frac{d^2y}{dt^2} = 6y^2 + t, \tag{1.2}$$

$$\mathrm{P_{II}}: \frac{d^2y}{dt^2} = 2y^3 + ty + \alpha, \tag{1.3}$$

$$\mathrm{P_{III}}: \frac{d^2y}{dt^2} = \frac{1}{y}\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} + \frac{\alpha y^2 + \beta}{t} + \gamma y^3 + \frac{\delta}{y}, \tag{1.4}$$

$$\mathrm{P_{IV}}: \frac{d^2y}{dt^2} = \frac{1}{2y}\left(\frac{dy}{dt}\right)^2 + \frac{3}{2}y^3 + 4ty^2 + 2\left(t^2 - \alpha\right)y + \frac{\beta}{y}, \tag{1.5}$$

$$\begin{aligned}\mathrm{P_V}: \frac{d^2y}{dt^2} = &\left(\frac{1}{2y} + \frac{1}{y-1}\right)\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} + \frac{(y-1)^2}{t}\left(\alpha y + \frac{\beta}{y}\right)\\ &+ \frac{\gamma y}{t} + \frac{\delta y(y+1)}{y-1},\end{aligned} \tag{1.6}$$

$$\begin{aligned}\mathrm{P_{VI}}: \frac{d^2y}{dt^2} = &\frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt}\\ &+ \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left[\alpha + \frac{\beta t}{y^2} + \frac{\gamma(t-1)}{(y-1)^2} + \frac{\delta t(t-1)}{(y-t)^2}\right].\end{aligned} \tag{1.7}$$

The Painlevé equations also appear in the problem of the monodromy preserving deformation of linear differential equations. R. Fuchs considered the second order linear differential equation of Fuchsian type:

$$\frac{d^2\psi}{d\lambda^2} = p(t)\psi \tag{1.8}$$

with the four regular singular points, $\lambda = 0,\ 1,\ \infty,\ t$ and the apparent singularity $\lambda = y$. He proved that the sixth Painlevé equation, $P_{VI}$, describes the condition that the linear differential equation has a fundamental system of solutions whose monodromy is independent of a variable $t$ ([4]). A result obtained by R. Garnier is connected to the isomonodromic deformation of the second order linear differential equation with irregular singularities. He showed that the other five Painlevé equations, $P_I$, $P_{II}$, $P_{III}$, $P_{IV}$, $P_V$, are obtained from completely integrability conditions of extended systems of the linear differential equation ([6]). L. Schlesinger considered the isomonodromic deformation of the linear system of the first order differential equations with regular singularities:

$$\frac{d\Psi}{d\lambda} = \sum_{\nu=1}^{n} \frac{A_\nu}{\lambda - a_\nu}\Psi, \tag{1.9}$$

and obtained the system of nonlinear differential equations ([31]):

$$\frac{\partial A_\nu}{\partial a_\mu} = \frac{[A_\mu, A_\nu]}{a_\mu - a_\nu} \quad (\mu \neq \nu), \tag{1.10}$$

$$\frac{\partial A_\nu}{\partial a_\nu} = -\sum_{\kappa(\neq\nu)} \frac{[A_\kappa, A_\nu]}{a_\kappa - a_\nu}, \tag{1.11}$$

where

$$[A_\mu, A_\nu] = A_\mu A_\nu - A_\nu A_\mu.$$

This system is obtained from the complete integrability condition of the extended system of (1.9):

$$\frac{\partial\Psi}{\partial\lambda} = \sum_{\nu=1}^{n} \frac{A_\nu}{\lambda - a_\nu}\Psi, \tag{1.12}$$

$$\frac{\partial\Psi}{\partial a_\nu} = -\frac{A_\nu}{\lambda - a_\nu}\Psi. \tag{1.13}$$

M. Jimbo, T. Miwa and K. Ueno established a general theory of monodromy preserving deformation for the matrix system of first order linear ordinary differential equations with regular or irregular singularities:

$$\frac{d\Psi}{d\lambda} = A(\lambda)\Psi, \tag{1.14}$$

where

$$A(\lambda) = \sum_{\nu=1}^{n} \sum_{k=1}^{r_\nu} \frac{A_{\nu,k}}{(\lambda - a_\nu)^k} - \sum_{k=2}^{r_\infty} A_{\infty,k} \lambda^{k-2}. \tag{1.15}$$

They defined monodromy data as a set of Stokes multipliers, connection matrices and exponents of formal monodromy, and they provided that the generalized monodromy preserving deformation is the deformation that monodromy data of a fundamental system of solutions are preserved ([8]). M. Jimbo and T. Miwa presented the linear systems with $2 \times 2$ matrices that the Painlevé equations are obtained from the compatibility condition of them ([9]). This linear systems are called the Lax pairs for the Painlevé equations. When $A(\lambda)$ (1.15) has a pole of degree $r_\nu$ at $\lambda = a_\nu$, the equation (1.14) is said to have a singular point of Poincaré rank $r_\nu - 1$. We associate with each of $\lambda = a_\nu$ $(\nu = 1, \ldots, n)$ a natural number $r_\nu$ such that the Poincaré rank of $\lambda = a_\nu$ is given by $r_\nu - 1$. We also associate with $\lambda = \infty$ a natural number $r_\infty$ such that the Poincaré rank of $\lambda = \infty$ is given by $r_\infty - 1$. Then we can represent such a system of linear differential equations by the following symbol:

$$(r_1, r_2, \ldots, r_n, r_\infty). \tag{1.16}$$

The system of linear differential equations considered in the studies on the Schlesinger system (1.9) is of the type:

$$\underbrace{(1, 1, \ldots, 1)}_{n+1}. \tag{1.17}$$

By the use of this notation, the correspondence of the types of the linear system with $2 \times 2$ matrices to the types of the Painlevé equations is the following:

$$\mathrm{P_{VI}}: (1, 1, 1, 1), \tag{1.18a}$$
$$\mathrm{P_{V}}: (1, 1, 2), \tag{1.18b}$$
$$\mathrm{P_{IV}}: (1, 3), \tag{1.18c}$$
$$\mathrm{P_{III}}: (2, 2), \tag{1.18d}$$
$$\mathrm{P_{II}}: (4). \tag{1.18e}$$

## 1.2 Soliton systems

The soliton theory grew from the study of the Korteweg-de Vries (KdV) equation. N. J. Zabusky and M. D. Kruskal studied the behavior of the numerical solutions of the KdV equation. They found that the solitary wave solutions had behavior similar to the superposition principle, despite the fact that the waves themselves were highly nonlinear. They named such waves solitons ([36]). This result led C. S. Gardner, J. M. Greene, M. D. Kruskal and R. M. Miura to the discovery of the inverse scattering transform method to solve the initial value problems for the KdV equation ([5]). P. D. Lax showed that the KdV equation is equivalent to the isospectral integrability condition for pairs of linear operators, known as Lax pairs. If we introduce the differential operators,

$$L = \partial_x^2 + u, \tag{1.19}$$

$$B = \partial_x^3 + \frac{3}{2}u\partial_x + \frac{3}{4}\partial_x u, \tag{1.20}$$

then the inverse scattering scheme for the KdV equation is written by

$$L\psi = \lambda\psi, \tag{1.21}$$

$$\partial_t \psi = B\psi. \tag{1.22}$$

If the eigenvalue $\lambda$ is independent of $x$ and $t$, then the compatibility condition of the equations (1.21) and (1.22) yields

$$\partial_t L = [B, L], \tag{1.23}$$

which reduces to the KdV equation ([18]). An extension of the Lax equation was given by V. E. Zakharov and A. B. Shabat. They treated the following equation for linear differential operators:

$$\partial_y B - \partial_t C + [B, C] = 0, \tag{1.24}$$

where

$$B = \sum_{j=0}^{m} b_j \partial_x^j, \tag{1.25}$$

$$C = \sum_{j=0}^{n} c_j \partial_x^j. \tag{1.26}$$

The equation (1.24) is obtained from the compatibility condition of

$$\partial_t \psi = B\psi, \tag{1.27}$$
$$\partial_y \psi = C\psi. \tag{1.28}$$

By choosing suitable operators $B$ and $C$, we can obtain several soliton equations from the equation (1.24). If we put

$$B = \partial_x^3 + \frac{3}{2} u \partial_x + v, \tag{1.29}$$
$$C = \partial_x^2 + u, \tag{1.30}$$

then the KP equation is obtained. If we suppose that $u$ is independent of $y$, then the KP equation reduces to the KdV equation; see [37]. M. Sato constructed the KP hierarchy and the multi-component KP hierarchy that include the KP equation. He discovered that the solutions of the KP hierarchy constitute an infinite-dimensional Grassmann manifold. He believed that any integrable system constitutes a submanifold of this infinite-dimensional Grassmann manifold. The unified approach to integrability makes us understand algebraically and geometrically integrable systems with infinitely many degree of freedom and their solutions ([29, 30]). This approach is known as the Sato theory at present ([27]).

## 1.3 Relations between soliton systems and Painlevé equations

The Painlevé equations are treated in the research of the mathematical physics. It was found by T. T. Wu, B. M. McCoy, C. A. Tracy and E. Barouch that the correlation function for the two-dimensional Ising model in the scaling region satisfies $\mathrm{P_{III}}$ ([35]). In the soliton theory, it was demonstrated by M. J. Ablowitz and H. Segur that similarity solutions of the soliton equations satisfy the Painlevé equations ([2]). M. J. Ablowitz, A. Ramani and H. Segur conjectured that a nonlinear partial differential equation is solvable by the inverse scattering method only if every nonlinear ordinary differential equation obtained by exact reduction has the Painlevé property ([1]). The relation between the isomonodromic deformation and the isospectral one was discussed; see [3, 10, 32, 33]. M. Jimbo and T. Miwa described a procedure to reduce the isospectral deformation into the isomonodromic deformation consistently by using the $\tau$-function. Following the procedure, one can obtain not only the Painlevé equations themselves but also the Lax pairs of them. $P_{III}$ and $P_{IV}$ were obtained through the reduction from the Pohlmeyer-Lund-Regge equation and the nonlinear Schrödinger equation, respectively ([10]). M. Noumi and Y. Yamada introduced a Painlevé system associated with the affine root system of type $A_{n-1}^{(1)}$ including $P_{II}$ $(A_1^{(1)})$, $P_{IV}$ $(A_2^{(1)})$ and $P_V$ $(A_3^{(1)})$ ([26]). The systems are equivalent to similarity reductions of the $n$-reduced modified KP hierarchy. The coefficients of the Lax pair for the system of type $A_{n-1}^{(1)}$ are $n \times n$ matrices ([25]). The similarity reductions of the Drinfel'd-Sokolov hierarchies was investigated by T. Ikeda, S. Kakei and T. Kikuchi; see [11, 12, 13, 17]. As a consequence, $P_V$ can be obtained from the modified Yajima-Oikawa equation, and $P_{VI}$ with four parameters can be derived from the three-wave resonant system. In the papers, [13, 17], since they dealt with linear systems with $3 \times 3$ matrices, the coefficients of the Lax pair of which they obtained were also $3 \times 3$ matrices. They showed that the $2 \times 2$ linear system can be obtained from the $3 \times 3$ linear system by the method of using the Laplace transformation ([7, 23]).

## 1.4 Purposes

We aim to seek systems of the isospectral deformations that are directly reduced to the Lax pairs for the Painlevé equations. Specially, we deal with the linear systems with $2 \times 2$ matrices, since the types of singular points of the linear system with $2 \times 2$ matrices correspond to the types of the Painlevé equations. We hardly find out such the correspondence of the linear system with matrices of larger size to the Painlevé equations. Besides reductions of the anti-self-dual Yang-Mills equations to ordinary differential equations produce the Painlevé equations. The $2 \times 2$ linear system of the anti-self-dual Yang-Mills equations is also reduced to the Lax pairs for the Painlevé equations ([22]). Consequently many researchers treated the $2 \times 2$ linear systems in order to investigate the Painlevé equations. We intend to study the Painlevé equations by relating the properties of the soliton equations to that of the Painlevé equations. In order to construct the signpost of this approach, we try to formulate the holonomic deformation by using the Sato theory.

## 1.5 Results

In this paper, we propose an infinite-dimensional integrable hierarchy that is a source of the Lax pair with $2 \times 2$ matrices for $P_{VI}$. This hierarchy is an extension of the two-component KP hierarchy by using a time dependent spectral parameter. The extension means that the hierarchy restricted to be independent of the introduced time variable is equal to the usual two-component KP hierarchy. We consider specially the $(1,1)$-reduction of the two-component KP hierarchy which is known as the nonlinear Schrödinger hierarchy. We formulate the extended hierarchy by using the Sato-Wilson formalism. We skillfully define the wave function which is a normal solution of the linear system. This wave function is similar to the integrand of the Gauss's hypergeometric integral. This similarity might be related to the fact that the linear system governed by $P_{VI}$ for an appropriate parameter has a solution that satisfies the Gauss's hypergeometric equation.

Then we consider the holonomic deformation in the same way in the isospectral deformation. We construct a system of linear differential equations in the spectral parameter $\lambda$ by using the wave function in the extended hierarchy. In the Sato-Wilson formalism, the rules of the hierarchy are given by the Sato equation for the gauge operator. We give a condition of the linear system with the deformations by the Sato equation with respect to the spectral parameter. This linear system is of the type:

$$(1, 1, 1, \infty). \tag{1.31}$$

We obtain nonlinear systems that describe the condition of the complete integrability of the linear systems. If we reduce the type of the linear system (1.31) to the type (1.18a), then the infinite-dimensional system is reduced to a one-dimensional system. We show that $P_{VI}$ is obtained from this one-dimensional system.

We also present a unified approach to treat the other Painlevé equations from the usual two-component KP hierarchy. We explain the nonlinear Schrödinger hierarchy by using the Sato-Wilson formalism. In each subsection, we define the different wave functions which are similar to the integrand of the some degenerated hypergeometric integral. These definitions have no effect on the two-component KP hierarchy, but an effect on the holonomic deformations. Namely the different Sato equations with respect to the spectral parameter are defined. We construct systems of linear differential equations in the spectral parameter $\lambda$ by using each wave function. The linear systems which are obtained are of the types:

$$(1,1,\infty), \tag{1.32a}$$
$$(1,\infty), \tag{1.32b}$$
$$(2,\infty), \tag{1.32c}$$
$$(\infty). \tag{1.32d}$$

We obtain nonlinear systems that describe the condition of the complete integrability of the linear systems. If we assume the following reductions for the linear systems (1.32a):

$$(1,1,\infty) \to (1,1,2), \tag{1.33a}$$
$$(1,\infty) \to (1,3), \tag{1.33b}$$
$$(2,\infty) \to (2,2), \tag{1.33c}$$
$$(\infty) \to (4), \tag{1.33d}$$

then the infinite-dimensional systems is reduced to one-dimensional systems which yield the other Painlevé equations. It follows that the reductions of the nonlinear Schrödinger equation give rise to not only $P_{IV}$ (see [10]), but also $P_V$ and $P_{III}$.

## 1.6 Remarks

It is important to study the extension of the two-component KP hierarchy that we propose, because this system is a typical example of the non-isospectral deformation, and directly relates $P_{VI}$. Since the system is the non-isospectral deformation, the solutions of this hierarchy do not constitute the universal Grassmann manifold. We would like to know what the solutions of this hierarchy constitute. F. Nijhoff, A. Hone and N. Joshi presented that similarity reductions of a partial differential equation of Schwarzian type (SPDE) lead to $P_{VI}$ ([24]). They gave a Lax pair of $2 \times 2$ matrices type for the SPDE. Therefore we consider that there is the relation between the similarity reductions of the SPDE and our result. It is a future problem to find this relationship.

## 1.7 Construction

In Section 2, we construct an extension of the two-component KP hierarchy by employing the Sato-Wilson formalism. In Section 3, we consider the holonomic deformation based on this extended hierarchy and obtain the nonlinear system that describes the condition of this deformation. We present that the nonlinear system reduces to $P_{VI}$. In Section 4, we study the holonomic deformation that contains the two-component KP hierarchy and show that the nonlinear systems that describes the condition of this deformation reduce to the other Painlevé equations, $P_V$, $P_{IV}$, $P_{III}$ and $P_{II}$.

# 2 An extension of the two-component KP hierarchy

In this section, we construct "an extension of the $(1,1)$-reduction of the two-component KP hierarchy". We formulate this hierarchy by using the Sato-Wilson formalism. We obtain an integrable system from the Zakharov-Shabat system.

## 2.1 Pseudo-differential operator

The multi-component theory of the KP hierarchy is established in the paper, [29]. The $n$-component KP hierarchy is formulated by matrix pseudo-differential operators of size $n \times n$, instead of scalar ones used in the one-component hierarchy. We explain some notation about the matrix pseudo-differential operators of size $n \times n$.

The action of the differential operator $\partial_x$ on an $n \times n$ matrix $f(x)$ is

$$\partial_x f(x) = \frac{d}{dx} f(x).$$

The operator $\partial_x^{-1}$ is defined by

$$\partial_x \partial_x^{-1} = \partial_x^{-1} \partial_x \equiv 1.$$

Pseudo-differential operators are defined by using the operators $\partial_x$ and $\partial_x^{-1}$.

**Definition 1.** A pseudo-differential operator with matrix-coefficients of size $n \times n$ is a linear operator,

$$\mathcal{A} = \sum_m a_m(x) \partial_x^m,$$

where $a_m(x)$ is an $n \times n$ matrix-valued function of $x$.

A sum of pseudo-differential operators is defined in the usual way by collecting terms, and their product is defined by the following extension of Leibniz's rule,

$$\mathcal{A}\mathcal{B} = \sum_{m,n} a_m(x)\partial_x^m b_n(x)\partial_x^n = \sum_{m,n}\sum_{k=0}^{\infty} \binom{i}{k} a_m(x) b_n^{(m)}(x)\partial_x^{m+n-k},$$

where

$$\binom{i}{k} = \begin{cases} \frac{i(i-1)\ldots(i-k+1)}{k!} & (k \geq 1) \\ 1 & (k = 0). \end{cases}$$

We define the differential operator part of a pseudo-differential operator $\mathcal{A}$ by

$$(\mathcal{A})_+ = \sum_{m\geq 0} a_m(x)\partial_x^m.$$

A pseudo-differential operator possesses a unique inverse, denoted by $\mathcal{A}^{-1}$.

## 2.2 Sato Equation

In the Sato-Wilson formalism, a pseudo-differential operator called the gauge operator plays an essential role. The coefficients of the gauge operator are dependent variables in the soliton system. The condition of the isospectral deformation is given by the Sato equations that the gauge operator should satisfy.

We define the gauge operator of size $2 \times 2$ by

$$\mathcal{W} = I + \sum_{k=1}^{\infty} w_k \partial_x^{-k}, \tag{2.1}$$

whose $2 \times 2$ coefficients matrices $w_k$ $(k \geq 1)$ do not depend on the parameter $x$. This condition for the coefficients is equivalent to "the $(1, 1)$-reduction". The gauge operator $\mathcal{W}$ can be used to define the operator

$$\mathcal{U} = \mathcal{W}\sigma_3\mathcal{W}^{-1} = \sigma_3 + \sum_{k=1}^{\infty} u_k \partial_x^{-k}, \tag{2.2}$$

where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The first few $u_k$'s are

$$u_1 = [w_1, \sigma_3], \tag{2.3a}$$

$$u_2 = [w_2, \sigma_3] - [w_1, \sigma_3]w_1, \tag{2.3b}$$

$$u_3 = [w_3, \sigma_3] - [w_2, \sigma_3]w_1 - [w_1, \sigma_3]\left(w_2 - {w_1}^2\right). \tag{2.3c}$$

We introduce a differential operator

$$\mathcal{M} = -\gamma\partial_x + x\partial_x^2 + \sigma_3\left(-c\partial_x + \sum_{n=1}^{\infty} nt_n\partial_x^{n+1}\right). \tag{2.4}$$

By employing the gauge operator $\mathcal{W}$ and the differential operator $\mathcal{M}$, we define differential operators $\mathcal{B}$ and $\mathcal{B}_n$ ($n \geq 1$) by

$$\mathcal{B} = \left(\mathcal{W}\mathcal{M}\mathcal{W}^{-1}\right)_+ = -\sum_{k=0}^{\infty} R_k\partial_x^k, \tag{2.5}$$

$$\mathcal{B}_n = \left(\mathcal{W}\sigma_3\partial_x^n\mathcal{W}^{-1}\right)_+ = \sum_{k=0}^{n-1} u_{n-k}\partial_x^k + \sigma_3\partial_x^n \quad (n \geq 1). \tag{2.6}$$

where

$$R_0 = w_1 + cu_1 - \sum_{n=1}^{\infty} nt_n u_{n+1}, \tag{2.7a}$$

$$R_1 = \gamma I + c\sigma_3 - \sum_{n=1}^{\infty} nt_n u_n, \tag{2.7b}$$

$$R_2 = -xI - t_1\sigma_3 - \sum_{n=2}^{\infty} nt_n u_{n-1}, \tag{2.7c}$$

$$R_k = -(k-1)t_{k-1}\sigma_3 - \sum_{n=k}^{\infty} nt_n u_{n-k+1} \quad (k \geq 3). \tag{2.7d}$$

Matrix operators

$$W = I + \sum_{k=1}^{\infty} w_k(\lambda - t)^k, \tag{2.8}$$

$$U = \sigma_3 + \sum_{k=1}^{\infty} u_k(\lambda - t)^k, \tag{2.9}$$

$$M = I\left(-\frac{\gamma}{\lambda - t} + \frac{x}{(\lambda - t)^2}\right) + \sigma_3\left(-\frac{c}{\lambda - t} + \sum_{n=1}^{\infty} \frac{nt_n}{(\lambda - t)^{n+1}}\right), \tag{2.10}$$

$$B = -\sum_{k=0}^{\infty} \frac{R_k}{(\lambda - t)^k}, \tag{2.11}$$

$$B_n = \sum_{k=0}^{n-1} \frac{u_{n-k}}{(\lambda - t)^k} + \frac{\sigma_3}{(\lambda - t)^n} \quad (n \geq 1) \tag{2.12}$$

are obtained from the pseudo-differential operators by replacing $\partial_x$ with $(\lambda - t)^{-1}$. We assume that the matrix operators satisfy

$$\partial_t W = BW - WM, \tag{2.13}$$

$$\partial_{t_n} W = B_n W - W\frac{\sigma_3}{(\lambda - t)^n} \quad (n \geq 1), \tag{2.14}$$

which we call the Sato equation hereafter.

Let us now define a wave function.

**Definition 2.** A wave function $\Psi(\lambda)$ is defined by the following expression:

$$\Psi(\lambda) = W\Psi_0(\lambda), \tag{2.15}$$

where

$$\begin{aligned}\Psi_0(\lambda) = \lambda^{\alpha}(\lambda - 1)^{\beta}(\lambda - t)^{\gamma} \exp\left(\frac{x}{\lambda - t}\right) \\ \times \operatorname{diag}\left\{\lambda^{a}(\lambda - 1)^{b}(\lambda - t)^{c} \exp\left(\sum_{n=1}^{\infty} \frac{t_n}{(\lambda - t)^n}\right),\right. \\ \left.\lambda^{-a}(\lambda - 1)^{-b}(\lambda - t)^{-c} \exp\left(-\sum_{n=1}^{\infty} \frac{t_n}{(\lambda - t)^n}\right)\right\}.\end{aligned} \tag{2.16}$$

The elements of the wave function are similar to the integrand of the Gauss's hypergeometric integral:

$$ {}_2F_1(a,b;c;t) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \lambda^{b-1}(1-\lambda)^{c-b-1}(1-t\lambda)^{-a} d\lambda. \tag{2.17} $$

We note that the matrix-valued function $\Psi_0(\lambda)$ satisfies

$$ \partial_x \Psi_0(\lambda) = \frac{1}{\lambda - t}\Psi_0(\lambda), \tag{2.18} $$

$$ \partial_t \Psi_0(\lambda) = M\Psi_0(\lambda) = \mathcal{M}\Psi_0(\lambda), \tag{2.19} $$

$$ \partial_{t_n} \Psi_0(\lambda) = \frac{\sigma_3}{(\lambda - t)^n}\Psi_0(\lambda) = \sigma_3 \partial_x^n \Psi_0(\lambda) \quad (n \geq 1). \tag{2.20} $$

This leads to the following theorem:

**Theorem 1.** *If a matrix operator $W$ satisfies the Sato equation* (2.13) *and* (2.14), *then the wave function $\Psi(\lambda)$ which can be derived from $W$ satisfy the linear systems,*

$$ \partial_x \Psi(\lambda) = \frac{1}{\lambda - t}\Psi(\lambda), \tag{2.21} $$

$$ \partial_t \Psi(\lambda) = B\Psi(\lambda), \tag{2.22} $$

$$ \partial_{t_n} \Psi(\lambda) = B_n\Psi(\lambda) \quad (n \geq 1). \tag{2.23} $$

*Proof.* We have

$$ \begin{aligned} \partial_x \Psi(\lambda) &= \partial_x W \Psi_0(\lambda) \\ &= W \partial_x \Psi_0(\lambda) \\ &= W \frac{1}{\lambda - t}\Psi_0(\lambda) \\ &= \frac{1}{\lambda - t} W \Psi_0(\lambda) \\ &= \frac{1}{\lambda - t}\Psi(\lambda), \end{aligned} \tag{2.24} $$

since $\partial_x$ and $W$ are commutative. We find

$$ \begin{aligned} \partial_t \Psi(\lambda) &= \partial_t \left(W\Psi_0(\lambda)\right) \\ &= (\partial_t W)\Psi_0(\lambda) + W(\partial_t \Psi_0(\lambda)) \\ &= (\partial_t W)\Psi_0(\lambda) + WM\Psi_0(\lambda) \\ &= (\partial_t W + WM)\, W^{-1}\Psi(\lambda) \\ &= B\Psi(\lambda) \end{aligned} \tag{2.25} $$

by the Sato equation (2.13). We obtain

$$\begin{aligned}\partial_{t_n}\Psi(\lambda) &= \partial_{t_n}\left(W\Psi_0(\lambda)\right)\\ &= (\partial_{t_n}W)\Psi_0(\lambda) + W(\partial_{t_n}\Psi_0(\lambda))\\ &= (\partial_{t_n}W)\Psi_0(\lambda) + W\frac{\sigma_3}{(\lambda-t)^n}\Psi_0(\lambda)\\ &= \left(\partial_{t_n}W + W\frac{\sigma_3}{(\lambda-t)^n}\right)W^{-1}\Psi(\lambda)\\ &= B_n\Psi(\lambda)\end{aligned} \tag{2.26}$$

by the Sato equation (2.14). □

If we set $\mu = 1/(\lambda - t)$, then the linear systems (2.21) and (2.23) are the $(1,1)$-reduction of the two-component KP hierarchy itself. It goes without saying that the spectral parameter $\mu$ is independent of $t_n$ $(n \geq 1)$. The time dependent spectral parameter means that the spectral parameter $\mu$ depends on the time variable $t$, that is, $\partial_t\mu = \mu^2$.

The Sato equations also lead to the following theorem:

**Theorem 2.** *If a matrix operator $W$ satisfies the Sato equation* (2.13) *and* (2.14), *then the matrix operators $U$, $B$ and $B_n$ satisfy the Lax-type systems,*

$$\partial_t U = [B, U], \tag{2.27}$$

$$\partial_{t_n} U = [B_n, U] \quad (n \geq 1), \tag{2.28}$$

*and the Zakharov-Shabat systems,*

$$\partial_t B_n - \partial_{t_n} B + [B_n, B] = 0 \quad (n \geq 1), \tag{2.29}$$

$$\partial_{t_m} B_n - \partial_{t_n} B_m + [B_n, B_m] = 0 \quad (n, m \geq 1). \tag{2.30}$$

*Proof.* From the definition of the pseudo-differential operator $\mathcal{U}$ (2.2), we find

$$U = W\sigma_3 W^{-1}. \tag{2.31}$$

Therefore we have

$$\begin{aligned}\partial_t U - [B, U] &= \partial_t(W\sigma_3 W^{-1}) - [B, W\sigma_3 W^{-1}]\\ &= \left[\left(\partial_t W - BW + WM\right)W^{-1}, W\sigma_3 W^{-1}\right]\\ &= 0\end{aligned} \tag{2.32}$$

by the Sato equation (2.13). We obtain

$$\begin{aligned}\partial_{t_n}U-[B_n,U]&=\partial_{t_n}(W\sigma_3W^{-1})-[B_n,W\sigma_3W^{-1}]\\&=\left[\left(\partial_{t_n}W-B_nW+W\frac{\sigma_3}{(\lambda-t)^n}\right)W^{-1},W\sigma_3W^{-1}\right]\\&=0\end{aligned}\tag{2.33}$$

by the Sato equation (2.14). We find

$$\begin{aligned}&\partial_tB_n-\partial_{t_n}B+[B_n,B]\\&\quad=-\partial_t\left\{\left(\partial_{t_n}W-B_nW+W\frac{\sigma_3}{(\lambda-t)^n}\right)W^{-1}\right\}\\&\qquad+\partial_{t_n}\left\{\left(\partial_tW-BW+WM\right)W^{-1}\right\}\\&\qquad-\left[B_n,\left(\partial_tW-BW+WM\right)W^{-1}\right]\\&\qquad-\left[\left(\partial_{t_n}W-B_nW+W\frac{\sigma_3}{(\lambda-t)^n}\right)W^{-1},\left(\partial_tW+WM\right)W^{-1}\right]\\&\quad=0\end{aligned}\tag{2.34}$$

by the Sato equations (2.13) and (2.14). We have

$$\begin{aligned}&\partial_{t_m}B_n-\partial_{t_n}B_m+[B_n,B_m]\\&=-\partial_{t_m}\left\{\left(\partial_{t_n}W-B_nW+W\frac{\sigma_3}{(\lambda-t)^n}\right)W^{-1}\right\}\\&\quad+\partial_{t_n}\left\{\left(\partial_{t_m}W-B_mW+W\frac{\sigma_3}{(\lambda-t)^m}\right)W^{-1}\right\}\\&\quad-\left[B_n,\left(\partial_{t_m}W-B_mW+W\frac{\sigma_3}{(\lambda-t)^m}\right)W^{-1}\right]\\&\quad-\left[\left(\partial_{t_n}W-B_nW+W\frac{\sigma_3}{(\lambda-t)^n}\right)W^{-1},\left(\partial_{t_m}W+W\frac{\sigma_3}{(\lambda-t)^m}\right)W^{-1}\right]\\&=0\end{aligned}\tag{2.35}$$

by the Sato equation (2.14). □

The systems (2.30) are equal to the Zakharov-Shabat systems in the $(1,1)$-reduction of the two-component KP hierarchy. The system (2.29) is the additional one in the extended hierarchy. So new integrable systems are obtained from the system (2.29). Since the left-hand side of (2.29) with $n = 1$ is

$$
\begin{aligned}
&\partial_t B_1 - \partial_{t_1} B + [B_1, B] \\
&= \partial_t u_1 + \partial_{t_1} w_1 + c\partial_{t_1} u_1 - u_2 - [u_1, w_1] - \sum_{n=1}^{\infty} n t_n \left(\partial_{t_1} u_{n+1} - [u_1, u_{n+1}]\right) \\
&\quad - \sum_{n=1}^{\infty} \left(\partial_{t_1} u_n - [u_1, u_n] - [\sigma_3, u_{n+1}]\right) \sum_{k=1}^{\infty} \frac{(n+k-1)t_{n+k-1}}{(\lambda - t)^k},
\end{aligned}
\tag{2.36}
$$

we obtain a system

$$
\partial_t u_1 + \partial_{t_1} w_1 + c\partial_{t_1} u_1 - u_2 - [u_1, w_1] - \sum_{n=1}^{\infty} n t_n \left(\partial_{t_1} u_{n+1} - [u_1, u_{n+1}]\right) = 0, \tag{2.37a}
$$

$$
\partial_{t_1} u_n - [u_1, u_n] - [\sigma_3, u_{n+1}] = 0 \quad (n \geq 1). \tag{2.37b}
$$

If we set $t_n \equiv 0$ $(n \geq 2)$, then the system (2.37) reduces to

$$
\partial_t u_1 + \partial_{t_1} w_1 + c\partial_{t_1} u_1 - u_2 - [u_1, w_1] - t_1 \left(\partial_{t_1} u_2 - [u_1, u_2]\right) = 0, \tag{2.38a}
$$

$$
\partial_{t_1} u_1 - [u_1, u_1] - [\sigma_3, u_2] = 0. \tag{2.38b}
$$

If we introduce the following parameterizations for the matrices

$$
u_1 = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}, \tag{2.39a}
$$

$$
u_2 = \begin{pmatrix} -uv/2 & f \\ g & uv/2 \end{pmatrix}, \tag{2.39b}
$$

$$
w_1 = \begin{pmatrix} d+e & -u/2 \\ v/2 & d-e \end{pmatrix}, \tag{2.39c}
$$

then we obtain a system

$$\partial_{t_1} u - 2f = 0, \tag{2.40a}$$
$$\partial_{t_1} v + 2g = 0, \tag{2.40b}$$
$$\partial_t u - 2\,(1-c)\,f - t_1 \partial_{t_1} f + 2eu + t_1 u^2 v = 0, \tag{2.40c}$$
$$\partial_t v - 2\,(1+c)\,g - t_1 \partial_{t_1} g - 2ev - t_1 u v^2 = 0, \tag{2.40d}$$
$$\partial_{t_1} e - \frac{1}{2} uv = 0, \tag{2.40e}$$
$$\partial_{t_1} d = 0. \tag{2.40f}$$

*Remark 2.1.* We have formulated the hierarchy by using the pseudo-differential operators. We can also formulate that by using the difference operators (see [34]). If the gauge operator $\mathcal{W}$ does not depend on the parameter $\gamma$, then we have

$$e^{-\partial_\gamma} \Psi(\lambda) = \frac{1}{\lambda - t} \Psi(\lambda). \tag{2.41}$$

Therefore the difference operators are obtained from the pseudo-differential operators by replacing $\partial_x$ with $e^{-\partial_\gamma}$.

# 3 The extended two-component system and the sixth Painlevé equation

In this section, we consider the holonomic deformation based on the integrable system in the previous section. We give a condition of the linear system with the deformations by the Sato equation with respect to the spectral parameter. We show that the condition of the holonomic deformation is described by a system that reduces to $\mathrm{P_{VI}}$.

If we introduce a differential operator

$$\begin{aligned}\mathcal{V} = I\Big(&\alpha\left\{1+(t-1)\,\partial_x\right\}+\beta\,(1+t\partial_x)\\&+\gamma\left\{1+(2t-1)\,\partial_x+t\,(t-1)\,\partial_x^2\right\}\\&-x\left\{\partial_x+(2t-1)\partial_x^2+t(t-1)\partial_x^3\right\}\Big)\\+\sigma_3\Big(&a\left\{1+(t-1)\,\partial_x\right\}+b\,(1+t\partial_x)\\&+c\left\{1+(2t-1)\,\partial_x+t\,(t-1)\,\partial_x^2\right\}\\&-\sum_{n=1}^{\infty}nt_n\left\{\partial_x^n+(2t-1)\,\partial_x^{n+1}+t\,(t-1)\,\partial_x^{n+2}\right\}\Big),\end{aligned} \tag{3.1}$$

then the matrix-valued function $\Psi_0(\lambda)$ (2.16) fulfills

$$\frac{\lambda(\lambda-1)}{\lambda-t}\partial_\lambda\Psi_0(\lambda)=\mathcal{V}\Psi_0(\lambda). \tag{3.2}$$

By using the gauge operator $\mathcal{W}$ and the differential operator $\mathcal{V}$, we define a differential operator $\mathcal{C}$ by

$$\mathcal{C}=\left(\mathcal{W}\mathcal{V}\mathcal{W}^{-1}\right)_+=\sum_{k=0}^{\infty}c_k\partial_x^k, \tag{3.3}$$

where

$$
\begin{aligned}
c_0 = {} & (\alpha+\beta+\gamma)I + (2t-1)w_1 - t(t-1)(-2w_2 + {w_1}^2) \\
& + a\,\{\sigma_3 + (t-1)u_1\} + b\,(\sigma_3 + tu_1) \\
& + c\,\{\sigma_3 + (2t-1)u_1 + t(t-1)u_2\} \\
& - \sum_{n=1}^{\infty} nt_n\,\{u_n + (2t-1)u_{n+1} + t(t-1)u_{n+2}\}\,,
\end{aligned}
\tag{3.4a}
$$

$$
\begin{aligned}
c_1 = {} & (\alpha(t-1) + \beta t + \gamma(2t-1) - x)I + t(t-1)w_1 \\
& + a(t-1)\sigma_3 + bt\sigma_3 + c\,\{(2t-1)\sigma_3 + t(t-1)u_1\} \\
& - t_1\,\{\sigma_3 + (2t-1)u_1 + t(t-1)u_2\} \\
& - \sum_{n=2}^{\infty} nt_n\,\{u_{n-1} + (2t-1)u_n + t(t-1)u_{n+1}\}\,,
\end{aligned}
\tag{3.4b}
$$

$$
\begin{aligned}
c_2 = {} & (\gamma t(t-1) - (2t-1)x)I + ct(t-1)\sigma_3 \\
& - t_1\,\{(2t-1)\sigma_3 + t(t-1)u_1\} \\
& - 2t_2\,\{\sigma_3 + (2t-1)u_1 + t(t-1)u_2\} \\
& - \sum_{n=3}^{\infty} nt_n\,\{u_{n-2} + (2t-1)u_{n-1} + t(t-1)u_n\}\,,
\end{aligned}
\tag{3.4c}
$$

$$
\begin{aligned}
c_3 = {} & -t(t-1)xI - t_1 t(t-1)\sigma_3 - 2t_2\,\{(2t-1)\sigma_3 + t(t-1)u_1\} \\
& - 3t_3\,\{\sigma_3 + (2t-1)u_1 + t(t-1)u_2\} \\
& - \sum_{n=4}^{\infty} nt_n\,\{u_{n-3} + (2t-1)u_{n-2} + t(t-1)u_{n-1}\}\,,
\end{aligned}
\tag{3.4d}
$$

$$
\begin{aligned}
c_k = {} & -(k-2)t_{k-2}t(t-1)\sigma_3 - (k-1)t_{k-1}\,\{(2t-1)\sigma_3 + t(t-1)u_1\} \\
& - kt_k\,\{\sigma_3 + (2t-1)u_1 + t(t-1)u_2\} \\
& - \sum_{n=k+1}^{\infty} nt_n\,\{u_{n-k} + (2t-1)u_{n-k+1} + t(t-1)u_{n-k+2}\} \quad (k \geq 4).
\end{aligned}
\tag{3.4e}
$$

We introduce matrix operators

$$S = \frac{\alpha I + a\sigma_3}{\lambda} + \frac{\beta I + b\sigma_3}{\lambda - 1} + \frac{\gamma I + c\sigma_3}{\lambda - t} - \sum_{n=1}^{\infty} \frac{n t_n \sigma_3}{(\lambda - t)^{n+1}}, \tag{3.5}$$

$$A = \frac{\lambda - t}{\lambda(\lambda - 1)} \sum_{k=0}^{\infty} \frac{c_k}{(\lambda - t)^k}. \tag{3.6}$$

We note that

$$\partial_\lambda \Psi_0(\lambda) = S\Psi_0(\lambda). \tag{3.7}$$

We assume that the matrix operator $A$ satisfies the Sato equation with respect to the spectral parameter:

$$\partial_\lambda W = AW - WS. \tag{3.8}$$

This leads to the following theorem:

**Theorem 3.** *If a matrix operator $W$ satisfies the reduction condition* (3.8)*, then the wave function $\Psi(\lambda)$* (2.15) *satisfies the linear system*

$$\partial_\lambda \Psi(\lambda) = A\Psi(\lambda). \tag{3.9}$$

*Proof.* We have

$$\begin{aligned} \partial_\lambda \Psi(\lambda) &= \partial_\lambda \left(W\Psi_0(\lambda)\right) \\ &= \left(\partial_\lambda W\right)\Psi_0(\lambda) + W\left(\partial_\lambda \Psi_0(\lambda)\right) \\ &= \left(\partial_\lambda W\right)\Psi_0(\lambda) + WS\Psi_0(\lambda) \\ &= \left(\partial_\lambda W + WS\right)W^{-1}\Psi(\lambda) \\ &= A\Psi(\lambda) \end{aligned} \tag{3.10}$$

by the condition (3.8). □

The Sato equations also lead to the following theorem:

**Theorem 4.** *If a matrix operator $W$ satisfies the Sato equation* (2.13), (2.14) *and* (3.8)*, then the matrix operators $U$ and $A$ satisfy the Lax-type systems,*

$$\partial_\lambda U = [A, U], \tag{3.11}$$

*and the matrix operators $A$, $B$ and $B_n$ satisfy the Zakharov-Shabat type systems,*

$$\partial_t A - \partial_\lambda B + [A, B] = 0, \tag{3.12}$$

$$\partial_{t_n} A - \partial_\lambda B_n + [A, B_n] = 0 \quad (n \geq 1). \tag{3.13}$$

*Proof.* We have

$$\begin{aligned} \partial_\lambda U - [A, U] &= \partial_\lambda (W \sigma_3 W^{-1}) - [A, W \sigma_3 W^{-1}] \\ &= \left[ \left( \partial_\lambda W - AW + WM \right) W^{-1}, W \sigma_3 W^{-1} \right] \\ &= 0 \end{aligned} \tag{3.14}$$

by the Sato equation (3.8). We find

$$\begin{aligned} &\partial_t A - \partial_\lambda B + [A, B] \\ &\quad = -\partial_t \left\{ \left( \partial_\lambda W - AW + WS \right) W^{-1} \right\} \\ &\qquad + \partial_\lambda \left\{ \left( \partial_t W - BW + WM \right) W^{-1} \right\} \\ &\qquad - \left[ A, \left( \partial_t W - BW + WM \right) W^{-1} \right] \\ &\qquad - \left[ \left( \partial_\lambda W - AW + WS \right) W^{-1}, \left( \partial_t W + WM \right) W^{-1} \right] \\ &\quad = 0 \end{aligned} \tag{3.15}$$

by the Sato equations (2.13) and (3.8). We have

$$\begin{aligned} &\partial_{t_n} A - \partial_\lambda B_n + [A, B_n] \\ &\quad = -\partial_{t_n} \left\{ \left( \partial_\lambda W - AW + WS \right) W^{-1} \right\} \\ &\qquad + \partial_\lambda \left\{ \left( \partial_{t_n} W - B_n W + W \frac{\sigma_3}{(\lambda - t)^n} \right) W^{-1} \right\} \\ &\qquad - \left[ A, \left( \partial_{t_n} W - B_n W + W \frac{\sigma_3}{(\lambda - t)^n} \right) W^{-1} \right] \\ &\qquad - \left[ \left( \partial_\lambda W - AW + WS \right) W^{-1}, \left( \partial_{t_n} W + W \frac{\sigma_3}{(\lambda - t)^n} \right) W^{-1} \right] \\ &\quad = 0 \end{aligned} \tag{3.16}$$

by the Sato equations (2.14) and (3.8). □

If we introduce matrices

$$\begin{aligned}P &= -\sum_{l=0}^{\infty} \frac{c_l}{(-t)^{l-1}} \\ &= \alpha I + t^2 w_1 - t^2(t-1)\left(-2w_2 + {w_1}^2\right) \\ &\quad + a(\sigma_3 + t(t-1)u_1) + bt^2 u_1 + ct^2(u_1 + (t-1)u_2) \\ &\quad - t^2 \sum_{n=1}^{\infty} n t_n \left(u_{n+1} + (t-1)u_{n+2}\right),\end{aligned} \tag{3.17a}$$

$$\begin{aligned}Q &= \sum_{l=0}^{\infty} \frac{c_l}{(1-t)^{l-1}} \\ &= \beta I - (t-1)^2 w_1 + t(t-1)^2\left(-2w_2 + {w_1}^2\right) \\ &\quad - a(t-1)^2 u_1 + b(\sigma_3 - t(t-1)u_1) - c(t-1)^2(u_1 + tu_2) \\ &\quad + (t-1)^2 \sum_{n=1}^{\infty} n t_n \left(u_{n+1} + t u_{n+2}\right),\end{aligned} \tag{3.17b}$$

then the relation

$$\frac{P}{t} + \frac{Q}{t-1} - R_0 = \frac{\alpha I + a\sigma_3}{t} + \frac{\beta I + b\sigma_3}{t-1} \tag{3.18}$$

holds. Then we have

$$A = \frac{P}{\lambda} + \frac{Q}{\lambda - 1} + \sum_{k=1}^{\infty} \frac{R_k}{(\lambda - t)^k}, \tag{3.19}$$

where the matrices $R_k$ are given by (2.7). If we put $t_n \equiv 0$ ($n \geq l$), then we have $R_k \equiv 0$ ($k \geq l+1$), and $A$ has a pole of degree $l$ at $\lambda = t$. In this case, the linear system (3.9) is said to have an irregular singular point at $\lambda = t$ of Poincaré rank $l-1$.

By using (2.11) and (3.19), the left-hand side of the system (3.12) becomes

$$\begin{aligned}&\partial_t A - \partial_\lambda B + [A, B] \\ &= \left(\partial_t P - \left[P, \sum_{l=0}^{\infty} \frac{R_l}{(-t)^l}\right]\right)\frac{1}{\lambda} + \left(\partial_t Q - \left[Q, \sum_{l=0}^{\infty} \frac{R_l}{(1-t)^l}\right]\right)\frac{1}{\lambda - 1} \\ &\quad + \sum_{k=1}^{\infty}\left(\partial_t R_k + [R_0, R_k] + \sum_{l=k}^{\infty}\left[\frac{P}{(-t)^{l-k+1}} + \frac{Q}{(1-t)^{l-k+1}}, R_l\right]\right)\frac{1}{(\lambda - t)^k}.\end{aligned} \tag{3.20}$$

It follows that we obtain the systems

$$\partial_t P - \left[P, \sum_{l=0}^{\infty} \frac{R_l}{(-t)^l}\right] = 0, \tag{3.21a}$$

$$\partial_t Q - \left[Q, \sum_{l=0}^{\infty} \frac{R_l}{(1-t)^l}\right] = 0, \tag{3.21b}$$

$$\partial_t R_k + [R_0, R_k] + \sum_{l=k}^{\infty} \left[\frac{P}{(-t)^{l-k+1}} + \frac{Q}{(1-t)^{l-k+1}}, R_l\right] = 0 \quad (k \geq 1). \tag{3.21c}$$

If we put $t_n \equiv 0\,(n \geq 1)$, $x \equiv 0$, then the coefficient matrices reduce to $R_1 = \gamma I + c\sigma_3$, $R_k \equiv 0\,(k \geq 2)$ and we have

$$\partial_t P - \left[P, R_0 - \frac{R_1}{t}\right] = 0, \tag{3.22a}$$

$$\partial_t Q - \left[Q, R_0 + \frac{R_1}{1-t}\right] = 0. \tag{3.22b}$$

We derive $\mathrm{P_{VI}}$ from the system (3.22). We use the following parameterization for the matrices $P$, $Q$:

$$P = \begin{pmatrix} p_0 + p_3 & p_{12} \\ p_{21} & p_0 - p_3 \end{pmatrix}, \tag{3.23a}$$

$$Q = \begin{pmatrix} q_0 + q_3 & q_{12} \\ q_{21} & q_0 - q_3 \end{pmatrix}. \tag{3.23b}$$

Because we find $\partial_t p_0 = \partial_t q_0 = 0$ and $\partial_t \det P = \partial_t \det Q = \partial_t \det(P + Q + R_0) = 0$, we will put

$$p^2 = \det P, \tag{3.24a}$$

$$q^2 = \det Q, \tag{3.24b}$$

$$n^2 = \det(P + Q + R_0). \tag{3.24c}$$

If we set

$$\frac{k(\lambda - y)}{\lambda(\lambda - 1)} = \frac{p_{12}}{\lambda} + \frac{q_{12}}{\lambda - 1}, \tag{3.25}$$

then we have

$$p_{12} = ky, \quad q_{12} = k(1-y). \tag{3.26}$$

If we put

$$z = \frac{p_0 + p_3}{y} + \frac{q_0 + q_3}{y-1} + \frac{-s+c}{y-t}, \tag{3.27}$$

then we find

$$\begin{aligned} p_3 = &-\frac{y}{2c(y-t)^2}\Big\{y(y-1)(y-t)^2 z^2 \\ &- 2(y-t)\left(p_0(y-1)(y-t) + q_0 y(y-t) + (y-1)(-sy+ct)\right) z \\ &+ (c-s)\left((2p_0(t-1) + 2q_0 t - s(2t-1) + c)(y-t) + (c-s)t(t-1)\right)\Big\} \\ &- p_0 + \frac{1}{2c}\left(p^2 - q^2\frac{y}{y-1} - n^2 y\right), \end{aligned} \tag{3.28a}$$

$$\begin{aligned} q_3 = &\frac{y-1}{2c(y-t)^2}\Big\{y(y-1)(y-t)^2 z^2 \\ &- 2(y-t)\left(p_0(y-1)(y-t) + q_0 y(y-t) + y(-s(y-1) + c(t-1))\right) z \\ &+ (c-s)\left((2p_0(t-1) + 2q_0 t - s(2t-1) - c)(y-t) + (c-s)t(t-1)\right)\Big\} \\ &- q_0 + \frac{1}{2c}\left(-p^2\frac{y-1}{y} + q^2 + n^2(y-1)\right). \end{aligned} \tag{3.28b}$$

The integrability condition (3.12) yields

$$\begin{aligned} &\frac{p_{12}'}{\lambda} + \frac{q_{12}'}{\lambda-1} - 2\left(\frac{p_3}{\lambda} + \frac{q_3}{\lambda-1} + \frac{c}{\lambda-t}\right)\left(\frac{p_{12}}{t} + \frac{q_{12}}{t-1}\right) \\ &\qquad + 2\left(\frac{p_{12}}{\lambda} + \frac{q_{12}}{\lambda-1}\right)\left(\frac{p_3-a}{t} + \frac{q_3-b}{t-1} + \frac{c}{\lambda-t}\right) = 0, \end{aligned} \tag{3.29a}$$

$$\frac{p_0' + p_3'}{y} + \frac{q_0' + q_3'}{y-1} + \left(\frac{p_{21}}{y} + \frac{q_{21}}{y-1}\right)\left(\frac{p_{12}}{t} + \frac{q_{12}}{t-1}\right) = 0. \tag{3.29b}$$

Using (3.29), we have

$$\frac{dy}{dt} = \frac{2y(y-1)(y-t)}{t(t-1)}\left(z - \frac{p_0}{y} - \frac{q_0}{y-1} + \frac{s}{y-t}\right), \tag{3.30a}$$

$$\begin{aligned}\frac{dz}{dt} &= \frac{1}{t(t-1)}\big\{(-3y^2 + 2(1+t) - t)z^2 \\ &\quad + 2((2y-1-t)p_0 + (2y-t)q_0 - (2y-1)s)z - n^2\big\} \\ &\quad - \frac{(c-s)(c+s-1)}{(y-t)^2} + \frac{p^2}{(t-1)y^2} - \frac{q^2}{t(y-1)^2},\end{aligned} \tag{3.30b}$$

$$\begin{aligned}\frac{d}{dt}\log k &= -\frac{2}{t(t-1)}\left(y(y-1)z - p_0(y-1) - q_0 y + c(y-t) - a(t-1) - bt\right) \\ &\quad + \frac{2(c-s)y(y-1)}{t(t-1)(y-t)}.\end{aligned} \tag{3.30c}$$

From the differential equations (3.30a) and (3.30b), we obtain $\mathrm{P_{VI}}$:

$$\begin{aligned}\frac{d^2y}{dt^2} &= \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt} \\ &\quad + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left[\kappa_1 + \frac{\kappa_2 t}{y^2} + \frac{\kappa_3(t-1)}{(y-1)^2} + \frac{\kappa_4 t(t-1)}{(y-t)^2}\right],\end{aligned} \tag{3.30d}$$

$$\begin{aligned}&\kappa_1 = 2\left((p_0 + q_0 - s)^2 - n^2\right), \quad \kappa_2 = -2\left({p_0}^2 - p^2\right), \\ &\kappa_3 = 2\left({q_0}^2 - q^2\right), \quad \kappa_4 = 2c(1-c).\end{aligned} \tag{3.30e}$$

# 4 The two-component KP hierarchy and the other Painlevé equations

In this section, we study the holonomic deformations that relates to the $(1,1)$-reduction of the two-component KP hierarchy. We obtain the systems that describe the condition of these deformations. We show that each system reduces to each Painlevé equation, $\mathrm{P_V}$, $\mathrm{P_{IV}}$, $\mathrm{P_{III}}$ and $\mathrm{P_{II}}$.

## 4.1 The fifth Painlevé equation

We explain the $(1,1)$-reduction of the two-component KP hierarchy. We show that the systems that describes the condition of the holonomic deformation that contains this hierarchy as a part reduces to $\mathrm{P_V}$. Therefore we find that $\mathrm{P_V}$ is obtained through the reduction from the nonlinear Schrödinger equation.

We define the gauge operator

$$\mathcal{W} = I + \sum_{k=1}^{\infty} w_k \partial_x^{-k} \tag{4.1}$$

whose $2 \times 2$ coefficients matrices $w_k$ do not depend on the parameter $x$. This condition for the coefficients is equivalent to "the $(1,1)$-reduction". By using the gauge operator $\mathcal{W}$, we define a pseudo-differential operator $\mathcal{U}$ by

$$\mathcal{U} = \mathcal{W}\sigma_3\mathcal{W}^{-1} = \sigma_3 + \sum_{k=1}^{\infty} u_k \partial_x^{-k}. \tag{4.2}$$

We define a differential operator $\mathcal{B}_n$ by

$$\mathcal{B}_n = \left(\mathcal{W}\sigma_3\partial_x^n\mathcal{W}^{-1}\right)_+ = \sum_{k=0}^{n-1} u_{n-k}\partial_x^k + \sigma_3\partial_x^n \quad (n \geq 1). \tag{4.3}$$

Matrix operators

$$W = I + \sum_{k=1}^{\infty} w_k \lambda^{-k}, \tag{4.4}$$

$$U = \sigma_3 + \sum_{k=1}^{\infty} u_k \lambda^{-k}, \tag{4.5}$$

$$B_n = \sum_{k=0}^{n-1} u_{n-k}\lambda^k + \sigma_3\lambda^n \quad (n \geq 1) \tag{4.6}$$

are obtained from the pseudo-differential operators by replacing $\partial_x$ with $\lambda$. We assume that the matrix operators satisfy the Sato equation

$$\partial_{t_n} W = B_n W - W\sigma_3\lambda^n \quad (n \geq 1). \tag{4.7}$$

We define a wave function

$$\Psi(\lambda) = W\Psi_0(\lambda), \tag{4.8}$$

where

$$\Psi_0(\lambda) = \lambda^\alpha(\lambda-1)^\beta \exp(x\lambda)$$
$$\times \begin{pmatrix} \lambda^a(\lambda-1)^b \exp(\sum_{n=1}^\infty t_n\lambda^n) & 0 \\ 0 & \lambda^{-a}(\lambda-1)^{-b}\exp(-\sum_{n=1}^\infty t_n\lambda^n) \end{pmatrix}. \tag{4.9}$$

This definition of the wave function is slightly different from the usual one. The element of $\Psi_0(\lambda)$ is similar to the integrand of the integral representation of the confluent hypergeometric function:

$${}_1F_1(a;b;t) = \frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)} \int_0^1 \lambda^{a-1}(1-\lambda)^{b-a-1}e^{t\lambda}d\lambda. \tag{4.10}$$

The difference does not affect the soliton system, but affects the system of the holonomic deformation. We note that the matrix-valued function $\Psi_0(\lambda)$ satisfies

$$\partial_x\Psi_0(\lambda) = \lambda\Psi_0(\lambda), \tag{4.11}$$
$$\partial_{t_n}\Psi_0(\lambda) = \sigma_3\lambda^n\Psi_0(\lambda) = \sigma_3\partial_x^n\Psi_0(\lambda) \quad (n \geq 1). \tag{4.12}$$

This leads to the following proposition:

**Proposition 5.** *If a matrix operator $W$ satisfies the Sato equation* (4.7)*, then the matrix operators $U$ and $B_n$ satisfy*

$$\partial_{t_n}U = [B_n, U] \quad (n \geq 1), \tag{4.13}$$
$$\partial_{t_m}B_n - \partial_{t_n}B_m + [B_n, B_m] = 0 \quad (n, m \geq 1). \tag{4.14}$$

*Furthermore, the wave function $\Psi(\lambda)$ satisfy the linear systems,*

$$\partial_x\Psi(\lambda) = \lambda\Psi(\lambda), \tag{4.15}$$
$$\partial_{t_n}\Psi(\lambda) = B_n\Psi(\lambda) \quad (n \geq 1). \tag{4.16}$$

If we choose $m = 1$ and $n = 2$, then the Zakharov-Shabat system (4.14)

$$\partial_{t_1}B_2 - \partial_{t_2}B_1 + [B_2, B_1] = 0 \tag{4.17}$$

yields

$$\partial_{t_1}u_1 + [u_2, \sigma_3] = 0, \tag{4.18a}$$
$$\partial_{t_1}u_2 - \partial_{t_2}u_1 + [u_2, u_1] = 0. \tag{4.18b}$$

If we use the parameterizations (2.39a) and (2.39b), then we have the nonlinear Schrödinger equation

$$\partial_{t_1} u - 2f = 0, \tag{4.19a}$$

$$\partial_{t_1} v + 2g = 0, \tag{4.19b}$$

$$\partial_{t_1} f - \partial_{t_2} u - u^2 v = 0, \tag{4.19c}$$

$$\partial_{t_1} g - \partial_{t_2} v + uv^2 = 0. \tag{4.19d}$$

We consider the holonomic deformation that contains the two-component system. If we introduce a differential operator

$$\begin{aligned} \mathcal{V} = {} & \alpha(-1+\partial_x) + \beta\partial_x + x(-\partial_x + \partial_x^2) \\ & + \sigma_3 \left\{ a(-1+\partial_x) + b\partial_x + \sum_{n=1}^{\infty} n t_n \left(-\partial_x^n + \partial_x^{n+1}\right) \right\}, \end{aligned} \tag{4.20}$$

then the matrix-valued function $\Psi_0(\lambda)$ (4.9) satisfies

$$\lambda(\lambda-1)\partial_\lambda \Psi_0(\lambda) = \mathcal{V}\Psi_0(\lambda). \tag{4.21}$$

By using the gauge operator $\mathcal{W}$ and the differential operator $\mathcal{V}$, we define a differential operator $\mathcal{C}$ by

$$\mathcal{C} = \left(\mathcal{W}\mathcal{V}\mathcal{W}^{-1}\right)_+ = \sum_{k=0}^{\infty} c_k e^{k\partial_s}, \tag{4.22}$$

where

$$c_0 = -\alpha I - w_1 + a(-\sigma_3 + u_1) + b u_1 + \sum_{n=1}^{\infty} n t_n (-u_n + u_{n+1}), \tag{4.23a}$$

$$c_1 = (\alpha + \beta - x)I + (a+b)\sigma_3 + t_1(-\sigma_3 + u_1) + \sum_{n=2}^{\infty} n t_n(-u_{n-1} + u_n), \tag{4.23b}$$

$$c_2 = xI + t_1\sigma_3 + 2t_2(-\sigma_3 + u_1) + \sum_{n=3}^{\infty} n t_n(-u_{n-2} + u_{n-1}), \tag{4.23c}$$

$$c_k = (k-1)t_{k-1}\sigma_3 + k t_k(-\sigma_3 + u_1) + \sum_{n=k+1}^{\infty} n t_n(-u_{n-k} + u_{n-k+1}) \; (k \geq 3). \tag{4.23d}$$

We introduce matrix operators

$$S = \frac{\alpha I + a\sigma_3}{\lambda} + \frac{\beta I + b\sigma_3}{\lambda - 1} + \sum_{n=1}^{\infty} n t_n \sigma_3 \lambda^{n-1}, \tag{4.24}$$

$$A = \frac{1}{\lambda(\lambda - 1)} \sum_{k=0}^{\infty} c_k \lambda^k. \tag{4.25}$$

We note that

$$\partial_\lambda \Psi_0(\lambda) = S\Psi_0(\lambda). \tag{4.26}$$

We assume that the matrix operator $A$ satisfies the condition

$$\partial_\lambda W = AW - WS. \tag{4.27}$$

This leads to the following proposition:

**Proposition 6.** *If a matrix operator $W$ satisfies the reduction condition* (4.27)*, then the matrix operators $U$, $A$ and $B_n$ satisfy*

$$\partial_\lambda U = [A, U], \tag{4.28}$$

$$\partial_{t_n} A - \partial_\lambda B_n + [A, B_n] = 0 \quad (n \geq 1). \tag{4.29}$$

*Furthermore, the wave function $\Psi(\lambda)$* (4.8) *satisfies the linear system,*

$$\partial_\lambda \Psi(\lambda) = A\Psi(\lambda). \tag{4.30}$$

If we introduce matrices

$$P = -c_0 = \alpha I + w_1 + a(\sigma_3 - u_1) - bu_1 + \sum_{n=1}^{\infty} n t_n (u_n - u_{n+1}), \tag{4.31a}$$

$$Q = \sum_{l=0}^{\infty} c_l = \beta I + b\sigma_3 - w_1 + (a+b)u_1 + \sum_{n=1}^{\infty} n t_n u_{n+1}, \tag{4.31b}$$

$$R_0 = \sum_{l=2}^{\infty} c_l = xI + t_1\sigma_3 + \sum_{n=2} n t_n u_{n-1}, \tag{4.31c}$$

$$R_k = \sum_{l=k+2}^{\infty} c_l = (k+1)t_{k+1}\sigma_3 + \sum_{n=k+2} n t_n u_{n-k-1} \quad (k \geq 1), \tag{4.31d}$$

then we have

$$A = \frac{P}{\lambda} + \frac{Q}{\lambda - 1} + \sum_{k=0}^{\infty} R_k \lambda^k. \tag{4.32}$$

By using (4.6) and (4.32), the left-hand side of the system (4.29) with $n = 1$ turns

$$\begin{aligned}
&\partial_{t_1} A - \partial_\lambda B_1 + [A, B_1] \\
&\quad = (\partial_{t_1} P + [P, u_1]) \frac{1}{\lambda} + (\partial_{t_1} Q + [Q, u_1 + \sigma_3]) \frac{1}{\lambda - 1} \\
&\qquad + \partial_{t_1} R_0 - \sigma_3 + [R_0, u_1] + [P + Q, \sigma_3] \\
&\qquad + \sum_{k=1}^{\infty} (\partial_{t_1} R_k + [R_k, u_1] + [R_{k-1}, \sigma_3]) \lambda^k.
\end{aligned} \tag{4.33}$$

Therefore we obtain the systems

$$\partial_{t_1} P + [P, u_1] = 0, \tag{4.34a}$$
$$\partial_{t_1} Q + [Q, u_1 + \sigma_3] = 0, \tag{4.34b}$$
$$\partial_{t_1} R_0 - \sigma_3 + [R_0, u_1] + [P + Q, \sigma_3] = 0, \tag{4.34c}$$
$$\partial_{t_1} R_k + [R_k, u_1] + [R_{k-1}, \sigma_3] = 0 \quad (k \geq 1). \tag{4.34d}$$

If we put $t_n \equiv 0\,(n \geq 2)$, then the coefficient matrices reduce to $R_0 = t_1\sigma_3$, $R_k \equiv 0$ $(k \geq 1)$, and then we have

$$\partial_{t_1} P + [P, u_1] = 0, \tag{4.35a}$$
$$\partial_{t_1} Q + [Q, u_1 + \sigma_3] = 0. \tag{4.35b}$$

This systems is equivalent to $\mathrm{P_V}$ in the paper, [9].

*Remark 4.1.* We can also formulate this hierarchy by using the difference operators ([34]). If the gauge operator $\mathcal{W}$ do not depend on the parameter $\alpha$, then we have

$$e^{\partial_\alpha} \Psi(\lambda) = \lambda \Psi(\lambda). \tag{4.36}$$

So the difference operators are obtained from the pseudo-differential operators by replacing $\partial_x$ with $e^{\partial_\alpha}$.

## 4.2 The fourth Painlevé equation

We consider the different holonomic deformation that relates to the hierarchy in the previous subsection. We show that the system that describes the deformation condition reduces to $\mathrm{P_{IV}}$. This fact follows the result in the paper, [10].

We employ the same soliton system in the previous subsection. But we define the wave function as follows:

$$\Psi(\lambda) = W\Psi_0(\lambda), \tag{4.37}$$

where

$$\Psi_0(\lambda) = \lambda^\alpha \exp(x\lambda) \begin{pmatrix} \lambda^a \exp\left(\sum_{n=1}^\infty t_n \lambda^n\right) & 0 \\ 0 & \lambda^{-a} \exp\left(-\sum_{n=1}^\infty t_n \lambda^n\right) \end{pmatrix}. \tag{4.38}$$

The element of $\Psi_0(\lambda)$ is similar to the integrand of the integral representation of the Hermite-Weber function:

$$H_\nu(t) = \frac{\Gamma(\nu+1)}{2\pi i} \int_C \lambda^{-\nu-1} e^{2t\lambda - \lambda^2} d\lambda. \tag{4.39}$$

The matrix-valued function $\Psi_0(\lambda)$ satisfies

$$\partial_x \Psi_0(\lambda) = \lambda \Psi_0(\lambda), \tag{4.40}$$
$$\partial_{t_n} \Psi_0(\lambda) = \sigma_3 \lambda^n \Psi_0(\lambda) = \sigma_3 \partial_x^n \Psi_0(\lambda) \quad (n \geq 1). \tag{4.41}$$

This leads to the following proposition:

**Proposition 7.** *If a matrix operator $W$ satisfies the Sato equation* (4.7)*, then the wave function $\Psi(\lambda)$ satisfy the linear systems,*

$$\partial_x \Psi(\lambda) = \lambda \Psi(\lambda), \tag{4.42}$$
$$\partial_{t_n} \Psi(\lambda) = B_n \Psi(\lambda) \quad (n \geq 1). \tag{4.43}$$

We present the reduction condition for the soliton system. If we introduce a differential operator

$$\mathcal{S} = I(\alpha + x\partial_x) + \sigma_3 \left( a + \sum_{n=1}^\infty t_n \partial_x^n \right), \tag{4.44}$$

then the matrix-valued function $\Psi_0(\lambda)$ (4.38) satisfies

$$\lambda\partial_\lambda \Psi_0(\lambda) = \mathcal{S}\Psi_0(\lambda). \tag{4.45}$$

By using the gauge operator $\mathcal{W}$ and the differential operator $\mathcal{S}$, we define a differential operator $\mathcal{A}$ by

$$\mathcal{A} = \left(\mathcal{W}\mathcal{S}\mathcal{W}^{-1}\right)_+ = \sum_{k=0}^{\infty} a_k \partial_x^k, \tag{4.46}$$

where

$$a_0 = \alpha I + a\sigma_3 + \sum_{n=1}^{\infty} n t_n u_n, \tag{4.47a}$$

$$a_1 = xI + t_1\sigma_3 + \sum_{n=2}^{\infty} n t_n u_{n-1}, \tag{4.47b}$$

$$a_k = k t_k \sigma_3 + \sum_{n=k+1}^{\infty} n t_n u_{n-k} \quad (k \geq 2). \tag{4.47c}$$

We introduce matrix operators

$$S = I(\alpha + x\lambda) + \sigma_3 \left( a + \sum_{n=1}^{\infty} t_n \lambda^n \right), \tag{4.48}$$

$$A = \sum_{k=0}^{\infty} a_k \lambda^k. \tag{4.49}$$

We assume that the matrix operator $A$ satisfies

$$\lambda\partial_\lambda W = AW - WS. \tag{4.50}$$

This leads to the following proposition:

**Proposition 8.** *If a matrix operator $W$ satisfies the reduction condition* (4.50)*, then the matrix operators $U$, $A$ and $B_n$ satisfy*

$$\lambda\partial_\lambda U = [A, U], \tag{4.51}$$

$$\partial_{t_n} A - \lambda\partial_\lambda B_n + [A, B_n] = 0 \quad (n \geq 1). \tag{4.52}$$

*Furthermore, the wave function $\Psi(\lambda)$* (4.37) *satisfies the linear system,*

$$\lambda\partial_\lambda \Psi(\lambda) = A\Psi(\lambda). \tag{4.53}$$

*Remark 4.2.* If we put $t_n \equiv 0$ $(n \geq l)$, then we have $a_k \equiv 0$ $(k \geq l)$. In this case, the linear system (4.53) has a regular singular point at $\lambda = 0$ and an irregular singular point at $\lambda = \infty$ of Poincaré rank $l - 1$. Hence we guess that the systems (4.52) are equivalent to the fourth Painlevé equation with several variables; see [14, 15, 16].

By using (4.6) and (4.49), the left-hand side of the system (4.52) with $n = 1$ turns

$$\begin{aligned} &\partial_{t_1} A - \lambda \partial_\lambda B_1 + [A, B_1] \\ &\qquad = \partial_{t_1} a_0 + [a_0, u_1] + (\partial_{t_1} a_1 - \sigma_3 + [a_1, u_1] + [a_0, \sigma_3])\, \lambda \\ &\qquad\quad + \sum_{k=2}^{\infty} (\partial_{t_1} a_k + [a_k, u_1] + [a_{k-1}, \sigma_3])\, \lambda^k. \end{aligned} \tag{4.54}$$

Hence we have the systems

$$\partial_{t_1} a_0 + [a_0, u_1] = 0, \tag{4.55a}$$

$$\partial_{t_1} a_1 - \sigma_3 + [a_1, u_1] + [a_0, \sigma_3] = 0, \tag{4.55b}$$

$$\partial_{t_1} a_k + [a_k, u_1] + [a_{k-1}, \sigma_3] = 0 \quad (k \geq 2). \tag{4.55c}$$

If we put $t_2 \equiv 1/2$, $t_n \equiv 0$ $(n \geq 3)$, then the coefficient matrices reduce to $a_2 = \sigma_3$, $a_k \equiv 0$ $(k \geq 3)$, and we have

$$\partial_{t_1} a_0 + [a_0, u_1] = 0, \tag{4.56a}$$

$$\partial_{t_1} a_1 - \sigma_3 + [a_1, u_1] + [a_0, \sigma_3] = 0. \tag{4.56b}$$

This systems is equivalent to $\mathrm{P_{IV}}$ in the paper, [9].

## 4.3 The third Painlevé equation

We present that the system that is the condition of the different holonomic deformation reduces to $\mathrm{P_{III}}$. So we find that $\mathrm{P_{III}}$ is obtained through the reduction from the nonlinear Schrödinger equation.

We employ the same soliton system in the previous subsection, and we give another reduction condition for the soliton system. If we introduce a differential operator

$$\mathcal{S} = I \left(\alpha \partial_x + x \partial_x^2\right) + \sigma_3 \left( a \partial_x + \sum_{n=1}^{\infty} n t_n \partial_x^{n+1} \right), \tag{4.57}$$

then the matrix-valued function $\Psi_0(\lambda)$ (4.38) satisfies

$$\lambda^2 \partial_\lambda \Psi_0(\lambda) = \mathcal{S} \Psi_0(\lambda). \tag{4.58}$$

By using the gauge operator $\mathcal{W}$ and the differential operator $\mathcal{S}$, we define a differential operator $\mathcal{A}$ by

$$\mathcal{A} = \left(\mathcal{W}\mathcal{S}\mathcal{W}^{-1}\right)_+ = \sum_{k=0}^{\infty} a_k \partial_x^k, \tag{4.59}$$

where

$$a_0 = -w_1 + a u_1 + \sum_{n=1}^{\infty} n t_n u_{n+1}, \tag{4.60a}$$

$$a_1 = \alpha I + a\sigma_3 + \sum_{n=1}^{\infty} n t_n u_n, \tag{4.60b}$$

$$a_2 = xI + t_1\sigma_3 + \sum_{n=2}^{\infty} n t_n u_{n-1}, \tag{4.60c}$$

$$a_k = (k-1) t_{k-1}\sigma_3 + \sum_{n=k}^{\infty} n t_n u_{n-k+1} \quad (k \geq 3). \tag{4.60d}$$

We introduce matrix operators

$$S = I\left(\alpha\lambda + x\lambda^2\right) + \sigma_3\left(a\lambda + \sum_{n=1}^{\infty} n t_n \lambda^{n+1}\right), \tag{4.61}$$

$$A = \sum_{k=0}^{\infty} a_k \lambda^k. \tag{4.62}$$

We assume that the matrix operator $A$ satisfies

$$\lambda^2 \partial_\lambda W = AW - WS. \tag{4.63}$$

This leads to the following proposition:

**Proposition 9.** *If a matrix operator $W$ satisfies the reduction condition* (4.63)*, then the matrix operators $U$, $A$ and $B_n$ satisfy*

$$\lambda^2 \partial_\lambda U = [A, U], \tag{4.64}$$

$$\partial_{t_n} A - \lambda^2 \partial_\lambda B_n + [A, B_n] = 0 \quad (n \geq 1). \tag{4.65}$$

*Furthermore, the wave function* $\Psi(\lambda)$ (4.37) *satisfies the linear system,*

$$\lambda^2\partial_\lambda\Psi(\lambda) = A\Psi(\lambda). \tag{4.66}$$

By using (4.6) and (4.62), the left-hand side of the system (4.65) with $n = 1$ is

$$\begin{aligned}\partial_{t_1}A - \lambda^2\partial_\lambda B_1 + [A, B_1]& \\ = \partial_{t_1}a_0 + [a_0, u_1] + (\partial_{t_1}a_1 + [a_1, u_1] + [a_0, \sigma_3])\,\lambda& \\ + (\partial_{t_1}a_2 - \sigma_3 + [a_2, u_1] + [a_1, \sigma_3])\lambda^2& \\ + \sum_{k=3}^{\infty}(\partial_{t_1}a_k + [a_k, u_1] + [a_{k-1}, \sigma_3])\lambda^k.&\end{aligned} \tag{4.67}$$

Thus we obtain the systems

$$\partial_{t_1}a_0 + [a_0, u_1] = 0, \tag{4.68a}$$
$$\partial_{t_1}a_1 + [a_1, u_1] + [a_0, \sigma_3] = 0, \tag{4.68b}$$
$$\partial_{t_1}a_2 - \sigma_3 + [a_2, u_1] + [a_1, \sigma_3] = 0, \tag{4.68c}$$
$$\partial_{t_1}a_k + [a_k, u_1] + [a_{k-1}, \sigma_3] = 0 \quad (k \geq 3). \tag{4.68d}$$

If we put $t_n \equiv 0$ $(n \geq 2)$, then the coefficient matrices reduce to $a_2 = t_1\sigma_3$, $a_k \equiv 0$ $(k \geq 3)$, and then we have

$$\partial_{t_1}a_0 + [a_0, u_1] = 0, \tag{4.69a}$$
$$\partial_{t_1}a_1 + [a_1, u_1] + [a_0, \sigma_3] = 0. \tag{4.69b}$$

We can obtain $\mathrm{P_{III}}$ from this system (4.69).

## 4.4 The second Painlevé equation

We present that the system that describes the condition of the different holonomic deformation reduces to $\mathrm{P_{II}}$.

We employ the same soliton system in Subsection 4.1. However we define the wave function as follows:

$$\Psi(\lambda) = W\Psi_0(\lambda), \tag{4.70}$$

where

$$\Psi_0(\lambda) = \lambda^\alpha e^{x\lambda}\begin{pmatrix}\exp\left(\sum_{n=1}^{\infty} t_n\lambda^n\right) & 0 \\ 0 & \exp\left(-\sum_{n=1}^{\infty} t_n\lambda^n\right)\end{pmatrix}. \tag{4.71}$$

Needless to say, the matrix-valued function $\Psi_0(\lambda)$ satisfies

$$\partial_x \Psi_0(\lambda) = \lambda \Psi_0(\lambda), \tag{4.72}$$
$$\partial_{t_n} \Psi_0(\lambda) = \sigma_3 \lambda^n \Psi_0(\lambda) = \sigma_3 \partial_x^n \Psi_0(\lambda) \quad (n \geq 1). \tag{4.73}$$

This leads to the following proposition:

**Proposition 10.** *If a matrix operator $W$ satisfies the Sato equation* (4.7)*, then the wave function $\Psi(\lambda)$ satisfy the linear systems,*

$$\partial_x \Psi(\lambda) = \lambda \Psi(\lambda), \tag{4.74}$$
$$\partial_{t_n} \Psi(\lambda) = B_n \Psi(\lambda) \quad (n \geq 1). \tag{4.75}$$

We give the reduction condition for the soliton system. If we introduce a differential operator

$$\mathcal{S} = I\left(\alpha \partial_x^{-1} + x\right) + \sigma_3 \sum_{n=1}^{\infty} n t_n \partial_x^{n-1}, \tag{4.76}$$

then the matrix-valued function $\Psi_0(\lambda)$ (4.71) satisfies

$$\partial_\lambda \Psi_0(\lambda) = \mathcal{S} \Psi_0(\lambda). \tag{4.77}$$

By using the gauge operator $\mathcal{W}$ and the differential operator $\mathcal{S}$, we define a differential operator $\mathcal{A}$ by

$$\mathcal{A} = \left(\mathcal{W} \mathcal{S} \mathcal{W}^{-1}\right)_+ = \sum_{k=0}^{\infty} a_k e^{k\partial_s}, \tag{4.78}$$

where

$$a_0 = xI + t_1 \sigma_3 + \sum_{n=2}^{\infty} n t_n u_{n-1}, \tag{4.79}$$

$$a_k = (k+1) t_{k+1} \sigma_3 + \sum_{n=k+2}^{\infty} n t_n u_{n-k-1} \quad (k \geq 1). \tag{4.80}$$

We introduce matrix operators

$$S = I(\alpha\lambda^{-1} + x) + \sigma_3 \sum_{n=1}^{\infty} n t_n \lambda^{n-1}, \tag{4.81}$$

$$A = \sum_{k=0}^{\infty} a_k \lambda^k. \tag{4.82}$$

We assume that the matrix operator $A$ satisfies

$$\partial_\lambda W = AW - WS. \tag{4.83}$$

This leads to the following proposition:

**Proposition 11.** *If a matrix operator $W$ satisfies the reduction condition* (4.83)*, then the matrix operators $U$, $A$ and $B_n$ satisfy*

$$\partial_\lambda U = [A, U], \tag{4.84}$$

$$\partial_{t_n} A - \partial_\lambda B_n + [A, B_n] = 0 \quad (n \geq 1). \tag{4.85}$$

*Furthermore, the wave function $\Psi(\lambda)$* (4.70) *satisfies the linear system,*

$$\partial_\lambda \Psi(\lambda) = A\Psi(\lambda). \tag{4.86}$$

*Remark 4.3.* If we put $t_n \equiv 0$ $(n \geq l)$, then we have $a_k \equiv 0$ $(k \geq l-1)$. In this case, the linear system (4.86) has an irregular singular point at $\lambda = \infty$ of Poincaré rank $l-1$. So we guess that the systems (4.85) are equivalent to the $A_g$-system; see [19, 20, 21].

By using (4.6) and (4.82), the left-hand side of the system (4.85) with $n = 1$ turns

$$\begin{aligned} &\partial_{t_1} A - \partial_\lambda B_1 + [A, B_1] \\ &\quad = \partial_{t_1} a_0 - \sigma_3 + [a_0, u_1] + \sum_{k=1}^{\infty} \left(\partial_{t_1} a_k + [a_k, u_1] + [a_{k-1}, \sigma_3]\right) \lambda^k. \end{aligned} \tag{4.87}$$

So we have the systems

$$\partial_{t_1} a_0 - \sigma_3 + [a_0, u_1] = 0, \tag{4.88a}$$

$$\partial_{t_1} a_k + [a_k, u_1] + [a_{k-1}, \sigma_3] = 0 \quad (k \geq 1). \tag{4.88b}$$

If we put $t_3 \equiv 1/3$, $t_n \equiv 0$ ($n = 2$, $n \geq 4$), then the coefficient matrices reduce to $a_2 = \sigma_3$, $a_k \equiv 0$ ($k \geq 3$), and we have

$$\partial_{t_1} a_0 - \sigma_3 + [a_0, u_1] = 0, \tag{4.89a}$$

$$\partial_{t_1} a_1 + [a_1, u_1] + [a_0, \sigma_3] = 0. \tag{4.89b}$$

This systems is equivalent to $\mathrm{P_{II}}$ in the paper, [9].

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